AN

ELEMENTARY TREATISE

ON

VARIABLE QUANTITIES

AN ELEMENTARY TREATISE ON

VARIABLE QUANTITIES

IN TWO PARTS:

THE DIRECT AND INVERSE

By HIRAM COOK

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Hiram Cook was born in Preston, New London County, Connecticut, on December 11, 1827, and died at Norwich, Connecticut, on May 26, 1917. This book, published after his death, stands witness to his lifelong love of mathematics and his desire to put his knowledge in this field at the service of others.

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PREFACE

The subject of this work is the same as that of the Differential and Integral Calculus, but parts of it are treated somewhat differently, especially the fundamental principles; and these, it is believed, are made so clear that any ordinary algebraic student can readily comprehend them.

In regard to a variable *quantity*, it is taken to mean as qualified—that is, its value is subject to a continual change, either increasing or decreasing. Now this being the case, it is evident that its value must have some rate of increase or decrease, uniform or variable, according to governing conditions. Upon this theory this work is founded, and it is hoped it so clears the way that it can be understandingly followed by those who are so inclined.

How is it in regard to a differential, so called, and the process of finding it? First an increment is added to the variable, and finally, in order to obtain what is sought, this increment is made equal to zero and to something at the same time—the *something* being taken as the differential of the variable. No wonder the student becomes nonplussed, for it is very difficult to conceive how even an infinitesimal, or "the last assignable value of a quantity" and zero can be identical. Being confronted by such a dilemma, he either has to accept the doctor's diagnosis or give the matter up in disgust.

Let it not be imagined that this work is claimed to be perfect by its author, or that he considers himself more than a tyro compared with the great mathematicians of the past or present. He simply gives his theory of the subject, believing it to be correct and both reasonable and comprehensible, and if approved, even by a few, he will not feel he has labored wholly in vain. It is a hard matter, however, to persuade a man to part with his idols; therefore, since *Infinitesimal* was born lang syne and has done good service, possibly it is unreasonable to expect that the little fellow should be summarily dismissed.

HIRAM COOK.

Norwich, Connecticut
1916

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PART ONE

DIRECT METHOD

DEFINITIONS

ART. 1. Two classes of quantities are employed: namely, constants and variables. Constants are usually represented by the first letters of the alphabet, *a, b, c,* etc., and variables by the last, *u, x, y,* etc.

The value of a constant remains the same throughout the same investigation; while that of a variable continually increases or decreases at either a uniform or variable rate.

2. The variable whose rate of increase or decrease is assumed to be uniform is called the independent variable, and the variable whose value depends on that of the independent variable is called the dependent variable. Thus u is the dependent and x the independent variable in

$$u = ax^2 + b.$$

3. The dependent variable is a function of the independent variable. Thus u is a function of x in

$$u = x^3 + ax + b,$$

which is expressed generally thus, $u = f(x)$, in which f is simply a symbol denoting function.

4. Functions are of two general classes, algebraic and transcendental.

A function is algebraic when the dependent variable equals the expression containing the independent variable in a purely algebraic form, as

$$u = a^2 - x^2.$$

A function is transcendental when the dependent variable equals the expression containing the independent variable in the form of an exponent, logarithm, sine, cosine, tangent, etc., as in

$$u = a^x;\ u = \log x;\ u = \sin x;\ u = \cos x;\ u = \tan x, \text{ etc.}$$

Transcendental functions are of two classes, logarithmic and circular.

5. Functions are also explicit, implicit, increasing, and decreasing.

An explicit function is one in which the dependent variable is directly expressed in terms of the independent variable, as in

$$u = ax^2 + b \text{ or } u = \log x.$$

An implicit function is one in which the value of the function is not directly expressed in terms of its variable and constants. Thus in the equation

$$y^2 + axy + bx^2 + c = 0$$

y is an implicit function of x—that is, y is not directly expressed in terms of x and the constants a, b, and c.

An increasing function is one in which the dependent variable will increase when the independent variable increases, or will decrease when the independent variable decreases, as in

$$u = ax^2 + b.$$

A decreasing function is one in which the dependent variable will increase when the independent variable decreases, or will decrease when the independent variable increases, as in

$$u = \frac{1}{x}.$$

6. A function may consist of two or more independent variables, as

$$u = ax \pm by \pm cz \text{ or } u = axyz.$$

7. The rate of a variable—that is, its rate of increase or decrease—is designated by writing d before it, as du represents the rate of u, dx of x, dy of y, etc.

8. A ratal coefficient is the rate of the dependent variable divided by that of the independent variable. Thus $\frac{du}{dx}$ is the ratal coefficient of $u = f(x)$.

Algebraic Functions

9. Illustrations of the application of the rates of variables.

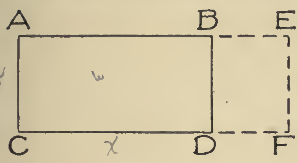


Fig. 1

Let the side AC of the rectangle $ABCD$ (Fig. 1) be represented by a, the side CD by x, and the area by $u = ax$. Extend AB to E, CD to F, and draw EF parallel to BD. Now let dx, the rate of increase of x, be represented by DF, and the area of $ABCD$ by $u = ax$; then adx will represent the area of $BEDF$, the rate of increase of $ABCD$, or du, the rate of u. Therefore the rate of

$$u = ax \tag{1}$$

is

$$du = adx = ax^{1-1}\, dx. \tag{2}$$

Extend AB of the rectangle $ABCD$ (Fig. 2) to K and G, also CD to L and H, and draw KL, EF, and GH parallel to AC. Now let AC be represented by a, FD by x, CF by y, and the area of $ABCD$ by $ax + ay$; also let dx, the rate of x, be represented by DH, and dy, the rate of y, by LC; then adx will represent the area of $BGDH$, the rate of increase of the area of $EBFD$, and ady will represent the area of $KALC$, the rate of increase of the area will represent the rate of increase of the area of $ABCD$, of $AECF$; hence $adx + ady$ or du, the rate of u. Therefore the rate of

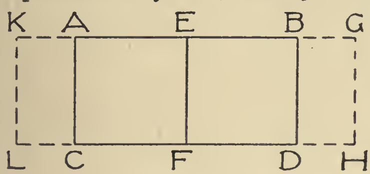


Fig. 2

$$u = ax + ay \tag{3}$$

is

$$du = adx + ady. \tag{4}$$

Extend the side AB of the rectangle $ABCD$ (Fig. 3) to G, CD to H, and draw EF, KL, and GH parallel to BD. Now let AC be represented by a, CD by x, FD by y, CF by $x - y$, and the area of $AECF$ by $u = ax - ay$; also let dx, the rate of x, be represented by DH, and dy, the rate of y, by DL; then adx will represent the area of $BGDH$, the rate of increase of $ABCD$, ady that of $BKDL$, the rate of increase of $EBFD$, and $adx - ady$, that of $KGLH$, the rate of increase of $ABCD$ less that of $EBFD$,

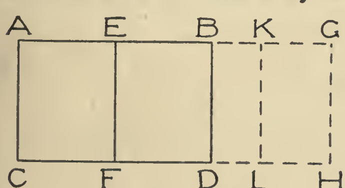


Fig. 3

or du, the rate of u. Therefore the rate of

$$u = ax - ay \tag{5}$$

is

$$du = adx - ady. \tag{6}$$

Extend the side AB of the rectangle $ABCD$ (Fig. 4) to G, CD to H, AC to E, BD to F, and draw EF parallel to AB, also GH to BD. Now let CD be represented by ax, AC by y, and the area of $ABCD$ by $u = axy$. Let adx, the rate of ax, be represented by DH, and dy, the rate of y, by AE; then $axdy$ will represent the area of $EFAB$, the rate of increase of the area of $ABCD$ in the direction of M, and $aydx$ will represent the area of $BGDH$, the rate of increase of the area of $ABCD$ in the direction of N. Hence $axdy + adyx$ will represent the total rate of increase of the area of $ABCD$, or du, the rate of u. Therefore the rate of

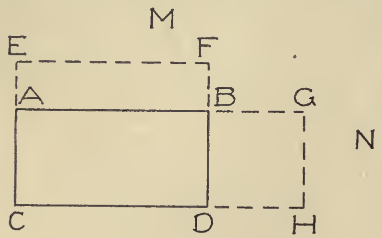


Fig. 4

$$u = axy \tag{7}$$

is

$$du = axdy + aydx. \tag{8}$$

10. Let $y = x$, then $dy = dx$. Substituting x for y in (7) of the last article, also x for y and dx for dy in (8), then

$$u = ax^2 \tag{1}$$

and

$$du = axdx + axdx = 2axdx = 2ax^{2-1}dx. \tag{2}$$

Let $y = x^2$, then $dy = 2xdx$. Substituting x^2 for y in (7) of the last article, also x^2 for y and $2xdx$ for dy in (8), then

$$u = ax^3 \tag{3}$$

and

$$du = 2ax^2dx + ax^2dx = 3ax^2dx = 3ax^{3-1}dx. \tag{4}$$

Let $y = x^3$, then $dy = 3x^2dx$. Substituting as before, the rate of

$$u = ax^4 \tag{5}$$

is

$$du = 4x^{4-1}dx. \tag{6}$$

Hence, if the exponent of x is n, n being a positive integer, from (1) of Art. 9 and from (2), (4), and (6) of the present article it is evident that the rate of

$$u = ax^n \tag{7}$$

is

$$du = anx^{n-1}dx. \tag{8}$$

When n is negative, as in

$$u = ax^{-n}, \tag{9}$$

multiplying both sides by x^n gives

$$ux^n = a.$$

Passing to the rate,

$$x^n du + nux^{n-1}dx = 0.$$

Transposing and substituting for u its value,

$$x^n du = -anx^{-1}dx,$$

and dividing by x^n, $du = -anx^{-n-1}dx.$ (10)

When the exponent of x is a positive fraction, as in

$$u = ax^{r/s}, \tag{11}$$

raising both sides of the equation to the sth power,

$$u^s = a^s x^r.$$

Passing to the rate,

$$su^{s-1}du = a^s r x^{r-1}dx. \tag{12}$$

Raising (11) to the $(s-1)$th power and multiplying by s,

$$su^{s-1} = a^{s-1} s x^{r-r/s}, \tag{13}$$

and dividing (12) by (13),

$$du = a\frac{r}{s}x^{r/s-1}dx. \tag{14}$$

When the exponent of x is a negative fraction, as in

$$u = ax^{-r/s}, \tag{15}$$

raising both sides of the equation to the sth power and multiplying by x^r give

$$u^s x^r = a^s.$$

Passing to the rate,

$$su^{s-1}x^r du + ru^s x^{r-1}dx = 0.$$

Transposing and dividing by $su^{s-1}x^r$ give

$$du = -\frac{r}{s}ux^{-1}dx,$$

or, since $u = ax^{-r/s}$,

$$du = -a\frac{r}{s}x^{-r/s-1}dx. \tag{16}$$

Hence, *the rate of a variable affected with any constant exponent, + or —, having also any constant coefficient, is the product of the coefficient and the exponent of the variable, multiplied by the variable with its exponent less unity, into the rate of the variable.*

EXAMPLES

1. $u = ax^{n+1}$
2. $u = ax^{3/4}$
3. $u = bx^{(n+1)/n}$
4. $u = \frac{1}{2} cx^{-2}$

11. To determine the rate of a function of the sum or difference of several independent variables, as

$$u = av + bx + cy + ez, \tag{1}$$

assume $u = r + s$, $r = av + bx$, and $s = cy + ez$, the rates of which [see (4) and (6) of Art. 10] are respectively

$$du = dr + ds, \; dr = adv + bdx, \text{ and } ds = cdy + edz. \tag{2}$$

Substituting the values of dr and ds in $du = dr + ds$ gives

$$du = adv + bdx + cdy + edz. \tag{3}$$

Hence *the rate of the sum or difference of several independent variables is the corresponding sum or difference of their rates taken separately.*

12. To determine the rate of

$$u = ax - bx^2 + cx^n. \tag{1}$$

Assume $v = ax$, $y = bx^2$, and $z = cx^n$,

the rates of which are [see (2) of Art. 9, and (2) and (10) of Art. 10]

$$dv = adx, \; dy = 2bxdx, \text{ and } dz = cnx^{n-1}dx. \tag{2}$$

But, according to the assumption,

$$u = v - y + z,$$

the rate of which is [see (1), Art. 11]

$$du = dv - dy + dz; \tag{3}$$

therefore, substituting in (3) the values of dv, dy, and dz, then

$$du = adx - 2bxdx + cnx^{n-1}dx. \tag{4}$$

Hence it is evident that *the rate of the sum or difference of any number of terms containing the same independent variable is the corresponding sum or difference of their rates taken separately.*

13. Required the rate of

$$u = (ax \pm by)^n. \qquad (1)$$

Assume $$u = v^n; \qquad (2)$$

then $$v = ax \pm by. \qquad (3)$$

Now the rate of (2), from Art. 10 is

$$du = nv^{n-1}dv, \qquad (4)$$

and the rate of (3) [see (3) and (5), Art. 9] is

$$dv = adx \pm bdy.$$

But $v^{n-1} = (ax \pm by)^{n-1}$, therefore, by substituting in (4) the values of v^{n-1} and dv, the result is

$$du = n\,(ax + by)^{n-1}\,(adx + bdy) \qquad (5)$$

or $$du = n\,(ax + by)^{n-1}adx + n\,(ax + by)^{n-1}bdy. \qquad (6)$$

Hence, *the rate of the nth power of the sum or difference of two variables, is n times their sum or difference raised to the $(n - 1)$th power, multiplied by the sum or difference of their rates, whether n be an integer or fraction, positive or negative.*

EXAMPLES

1. $u = x^2 - \frac{1}{4}x + 2x^{3/2}$ 2. $u = x^n + ax^{-r} + b$

3. $u = ax^{1-n} + nx^{1/n}$ 4. $u = (x + ay)^{n+1}$

14. To determine the rate of

$$u = vxy.$$

Assume $$z = xy; \qquad (1)$$

then $$u = vz,$$

and the rates of these, from Art. 9, are

$$dz = xdy + ydx \qquad (2)$$

and $$du = vdz + zdv. \qquad (3)$$

Substituting the value of z from (1), and dz from (2), in (3), then

$$du = vxdy + vydx + xydv. \qquad (4)$$

Hence it is evident that *the rate of the product of any number of variables is the sum of the products obtained by multiplying the rate of each variable by the product of the others.*

15. To determine the rate of

$$u = x^r (a + x)(bx^2 + cx^n).$$

Assume $$v = a + x \qquad (1)$$

and $$y = bx^2 + cx^n; \qquad (2)$$

then $$u = x^r vy.$$

Passing to the rate, (1) becomes

$$dv = dx, \qquad (3)$$

(2), by Art. 12,

$$dy = 2bxdx + cnx^{n-1}dx = (2bx + cnx^{n-1})\, dx, \qquad (4)$$

and $u = x^r vy$, by Art. 14,

$$du = x^r vdy + x^r ydv + rx^{r-1} vydx. \qquad (5)$$

Substituting the values of v and y from (1) and (2), also the values of dv and dy from (3) and (4), in (5), the result is

$$du = x^r (a + x)(2bx + cnx^{n-1})\, dx + x^r (bx^2 + cx^n)\, dx +$$
$$rx^{r-1} (a + x)(bx^2 + cx^n)\, dx,$$

or $$du = \{x^r (a + x)(2bx + cnx^{n-1}) +$$
$$x^r (bx^2 + cx^n) + rx^{r-1} (a + x)(bx^2 + cx^n)\} dx.$$

Hence, *the rate of the product of any number of factors containing the same variable is the sum of the products obtained by multiplying the rate of each factor by the product of the others.*

16. To determine the rate of

$$u = x^2 (a - z^n)(b + y^r).$$

Assume $$v = a - z^n \qquad (1)$$

and $$w = b + y^r, \qquad (2)$$

then $$u = x^2 vw. \qquad (3)$$

The rate of (1) is

$$dv = -nz^{n-1}dz, \tag{4}$$

that of (2)

$$dw = ry^{r-1}dy, \tag{5}$$

and that of (3)

$$du = x^2vdw + x^2wdv + 2xvwdx. \tag{6}$$

Substituting the values of v and w from (1) and (2), also the values of dv and dw from (4) and (5), in (6) gives

$$du = rx^2 (a - z^n) y^{r-1}dy - nx^2 (b + y^r) z^{n-1}dz + 2x (b + y^r) (a - z^n) dx.$$

Hence *the preceding rule is also applicable when each factor contains a different variable, or, as is evident, even when each factor contains several variables.*

EXAMPLES

1. $u = x^2yz^n$
2. $u = (bx + c) (x^n + ax)$
3. $u = x^2 (y^3 + av)$

17. To determine the rate of

$$u = (a + bx + cx^2)^n.$$

Assume

$$y = a + bx + cx^2; \tag{1}$$

then

$$u = y^n.$$

Passing to the rate,

$$dy = (b + 2cx) dx \tag{2}$$

and

$$du = ny^{n-1}dy. \tag{3}$$

Substituting the value of y from (1), and dy from (2), in (3), then

$$du = n (a + bx + cx^2)^{n-1} (b + 2cx) dx.$$

Hence, *the rate of a polynomial affected with any constant exponent is the exponent into the polynomial with its exponent less unity, multiplied by the rate of the polynomial.*

18. To determine the rate of

$$u = \sqrt{(ax + bx^n)}$$

or

$$u = (ax + bx^n)^{\frac{1}{2}}.$$

Passing to the rate, as in Art. 17,

$$du = \frac{1}{2}(ax + bx^n)^{-\frac{1}{2}}(a + nbx^{n-1})\,dx,$$

or $$du = \frac{(a + nbx^{n-1})\,dx}{2(ax + bx^n)^{\frac{1}{2}}} = \frac{(a + nbx^{n-1})\,dx}{2\sqrt{(ax + bx^n)}}.$$

Hence, *the rate of the square root of a quantity is the rate of the quantity under the radical, divided by twice the radical.*

19. To determine the rate of a fraction, as the function

$$u = \frac{v}{z}.$$

Multiplying through by z, $uz = v$;
then passing to the rate, by (7) of Art. 9,

$$udz + zdu = dv.$$

Substituting for u its value and transposing give

$$zdu = dv - \frac{vdz}{z}$$

or $$zdu = \frac{zdv - vdz}{z}.$$

Therefore, dividing by z,

$$du = \frac{zdv - vdz}{z^2}.$$

Hence *the rate of a fraction is the denominator into the rate of the numerator, minus the numerator into the rate of the denominator, divided by the square of the denominator.*

If v be a constant, then, since a constant has no rate,

$$du = -\frac{vdz}{z^2};$$

that is, when v is a constant, u is a decreasing function of z and its rate is consequently negative.

EXAMPLES

1. $u = \sqrt{(1 + x^2)}$
2. $u = \sqrt{(x^2 + y^2)}$
3. $u = x^n (x - a)(a - x^2)$
4. $u = \dfrac{\sqrt{(x+1)} + \sqrt{(x-1)}}{\sqrt{(x+1)} - \sqrt{(x-1)}}$

SUCCESSIVE RATES AND RATAL COEFFICIENTS

20. In obtaining these, and at the same time to exemplify the work, let

$$u = x^n + ax^2. \qquad (1)$$

Passing to the rate, by Art. 12,

$$du = (nx^{n-1} + 2ax)\, dx. \qquad (2)$$

Passing to the rate again, regarding dx as constant,

$$d\,(du) = d^2u = \{n\,(n-1)\,x^{n-2} + 2a\}\, dx^2. \qquad (3)$$

In like manner it will be found from (3) that

$$d^3u = n\,(n-1)\,(n-2)\,x^{n-3}dx^3. \qquad (4)$$

(2), (3), and (4) are successive rates of (1), and

$$(nx^{n-1} + 2ax), \{n\,(n-1)\,x^{n-2} + 2a\}$$

and $\{n\,(n-1)\,(n-2)\,x^{n-3}\}$ are respectively coefficients of dx, dx^2, and dx^3.

Dividing (2) by dx, (3) by dx^2, and (4) by dx^3, the results are

$$\frac{du}{dx} = nx^{n-1} + 2ax \qquad (5)$$

$$\frac{d^2u}{dx^2} = n\,(n-1)\,x^{n-2} + 2a \qquad (6)$$

and

$$\frac{d^3u}{dx^3} = n\,(n-1)\,(n-2)\,x^{n-3}. \qquad (7)$$

Inasmuch as $\frac{du}{dx}, \frac{d^2u}{dx^2}$, and $\frac{d^3u}{dx^3}$ are respectively equal to the coefficients of dx, dx^2, and dx^3, they are called ratal coefficients; du, d^2u, and d^3u are the first, second, and third rates of the dependent variable u, and dx, dx^2, and dx^3 are the first, second, and third powers of the rate of the independent variable x.

RATES OF FUNCTIONS OF TWO OR MORE INDEPENDENT VARIABLES

21. It has been shown in Art. 9 that the rate of $u = xy$ is $du = ydx + xdy$; therefore,

if $$u = x^n y,$$
its rate is $$du = nx^{n-1}y\,dx + x^n dy;$$
and if $$u = x^n y^m,$$
its rate is $$du = nx^{n-1}y^m dx + mx^n y^{m-1} dy;$$
also if $$u = x^n y^m + x^r y^s, \tag{1}$$
its rate is
$$du = nx^{n-1}y^m dx + mx^n y^{m-1} dy + rx^{r-1}y^s dx + sx^r y^{s-1} dy. \tag{2}$$

See preceding rules.

Now if the rates of (1) be first taken under the supposition that x varies and y remains constant, then that y varies and x remains constant, the sum of the results will be the same as (2): thus

$$du = nx^{n-1}y^m dx + rx^{r-1}y^s dx \tag{3}$$
and $$du = mx^n y^{m-1} dy + sx^r y^{s-1} dy. \tag{4}$$

Adding (3) and (4),
$$du = nx^{n-1}y^m dx + rx^{r-1}y^s dx + mx^n y^{m-1} dy + sx^r y^{s-1} dy. \tag{2}$$

Dividing (3) by dx and (4) by dy, the results are

$$\frac{du}{dx} = nx^{n-1}y^m + rx^{r-1}y^s \tag{5}$$
and $$\frac{du}{dy} = mx^n y^{m-1} + sx^r y^{s-1}. \tag{6}$$

By taking the rate of (5) with respect to y and the rate of (6) with respect to x, the following are found,

$$\frac{d^2u}{dxdy} = mnx^{n-1}y^{m-1} + rsx^{r-1}y^{s-1} \tag{7}$$
and $$\frac{d^2u}{dydx} = mnx^{n-1}y^{m-1} + rsx^{r-1}y^{s-1} \tag{8}$$

in which the right-hand members are identical; therefore

$$\frac{d^2u}{dxdy} = \frac{d^2u}{dydx}.$$

(2) is called the total rate, (3) and (4) partial rates, and (5) and (6) ratal coefficients. The second, third, and higher rates can be found in a similar manner as in Art. 20.

If the function contains three independent variables, as

$$u = f(x, y, z)$$

by proceeding in like manner the following results will be obtained:

$$\frac{d^2u}{dxdy} = \frac{d^2u}{dydx}, \quad \frac{d^2u}{dxdz} = \frac{d^2u}{dzdx}, \quad \frac{d^2u}{dydz} = \frac{d^2u}{dzdy}. \tag{9}$$

If the function contains four independent variables, there will be six of these equalities, if five there will be ten, and so on.

Special Rates

22. Let $$u = (x + y)^n, \tag{1}$$

then, by taking the rate first with respect to x and secondly with respect to y, the following are found,

$$du = n(x + y)^{n-1}dx \tag{2}$$

and $$du = n(x + y)^{n-1}dy. \tag{3}$$

Dividing the first by dx and the second by dy gives

$$\frac{du}{dx} = n(x + y)^{n-1} \tag{4}$$

and $$\frac{du}{dy} = n(x+y)^{n-1} \tag{5}$$

in which it will be observed that the right-hand members are identical.

The sum of (2) and (3) is

$$du = n(x + y)^{n-1}(dx + dy),$$

virtually the same as given in Art. 13.

Classified Rates

23. When the partial rates of a function of two or more independent variables are taken with respect to one variable only, they are said to be of the first class; when taken with respect to one variable and that rate taken with respect to another variable, they are said to be of the second class: thus, if

$$u = x^3y^2 + x^2y, \tag{1}$$

by taking the rate with respect to x only, the results are

$$du = 3x^2y^2dx + 2xydx$$

$$d^2u = 6xy^2dx^2 + 2ydx^2$$

and $$d^3u = 6y^2dx^3,$$

which are partial rates of the first class.

Taking the rate of (1) with respect to x gives

$$du = 3x^2y^2dx + 2xydx \tag{2}$$

and this rate taken with respect to y is

$$d^2u = 6x^2ydxdy + 2xdxdy; \tag{3}$$

(2) and (3) are partial rates of the second class.

Maclaurin's Theorem

This theorem explains the method of developing into a series a function of a single independent variable.

24. Assume the development to be

$$f(x) = A + Bx + Cx^2 + Dx^3 + \text{etc.}, \tag{1}$$

in which A, B, C, D, etc. are constants whose values depend entirely upon those which enter $f(x)$.

Now in order to determine the values of A, B, C, D, etc. such as will render the assumed development true for all values of x, let

$$u = A + Bx + Cx^2 + Dx^3 + \text{etc.} \tag{2}$$

and of this find the ratal coefficient, as in Art. 20; thus

$$\frac{du}{dx} = B + 2Cx + 3Dx^2 + \text{etc.} \tag{3}$$

$$\frac{d^2u}{dx^2} = 2C + 2 \cdot 3Dx + \text{etc.} \tag{4}$$

$$\frac{d^3u}{dx^3} = 2 \cdot 3D + \text{etc.} \tag{5}$$

Making $x = 0$, it will be found from (3), (4), (5), etc. that

$$B = \frac{du}{dx}, \quad C = \frac{1}{2} \cdot \frac{d^2u}{dx^2}, \quad D = \frac{1}{2 \cdot 3} \cdot \frac{d^3u}{dx^3}, \text{ etc.} \tag{6}$$

Since A will retain the same value, whatever the value of x, substituting the values of B, C, D, etc. in (2) will give

$$u = A + \frac{xdu}{dx} + \frac{x^2d^2u}{1 \cdot 2dx^2} + \frac{x^3d^3u}{1 \cdot 2 \cdot 3dx^3} + \text{etc.}, \tag{7}$$

the theorem of Maclaurin.

If the exponent of the variable is greater than unity, as $u = (a + bx^2)$, assume $bx^2 = v$; then substitute for v and its rates their values in the development of $u = (a + v)$.

When the function or any of its ratal coefficients becomes infinite by making its variable equal to zero, it can not be developed by this theorem—as, for instance, $u = ax^{1/2}$.

For an exemplification of this theorem, take

$$u = (a + x)^n. \qquad (1)$$

Determining the ratal coefficient, as in Art. 20,

$$\frac{du}{dx} = n\,(a + x)^{n-1} \qquad (2)$$

$$\frac{d^2u}{dx^2} = n\,(n - 1)\,(a + x)^{n-2} \qquad (3)$$

$$\frac{d^3u}{dx^3} = n\,(n - 1)\,(n - 2)\,(a + x)^{n-3}, \text{ etc.} \qquad (4)$$

Making $x = 0$, then from (2), (3), and (4) are found

$$\frac{du}{dx} = na^{n-1}$$

$$\frac{d^2u}{dx^2} = n\,(n - 1)\,a^{n-2}$$

$$\frac{d^3u}{dx^3} = n\,(n - 1)\,(n - 2)\,a^{n-3}$$

and from (1), when $x = 0$, $u = a^n$: that is, $A = a^n$.

Substituting these values in (7) will give

$$u = a^n + \frac{na^{n-1}x}{1} + \frac{n\,(n - 1)\,a^{n-2}x^2}{1 \cdot 2} + \frac{n\,(n - 1)\,(n - 2)\,a^{n-3}x^3}{1 \cdot 2 \cdot 3} + \text{etc.},$$

the same as found by the binomial theorem.

EXAMPLES

1. $u = (1 + x)^n$ 2. $u = (a + bx)^{-2}$

Taylor's Theorem

25. This theorem explains the method of developing into a series any function of the sum or difference of two independent variables, according to the ascending powers of one of them.

Let $$u = f(x+y) \qquad (1)$$
and assume the development to be
$$u = A + By + Cy^2 + Dy^3 + \text{etc.}, \qquad (2)$$
in which A, B, C, D, etc. are functions of x.

Now in order to find the values of A, B, C, D, etc., such as will render the development true for all possible values which may be ascribed to x and y, determine the ratal coefficients of (1), first under the supposition that x varies and y remains constant, then that y varies and x remains constant. By this process it will be found that
$$\frac{du}{dx} = \frac{dA}{dx} + \frac{dB}{dx}y + \frac{dC}{dx}y^2 + \frac{dD}{dx}y^3 + \text{etc.} \qquad (3)$$
and $$\frac{du}{dy} = B + 2Cy + 3Dy^2 + \text{etc.}, \qquad (4)$$
but since these ratal coefficients are identical [see (4) and (5) of Art. 22] it follows that
$$B + 2Cy + 3Dy^2 = \frac{dA}{dx} + \frac{dB}{dx}y + \frac{dC}{dx}y^2 + \frac{dD}{dx}y^3 + \text{etc.}, \qquad (5)$$
in which the coefficients of like powers of y must evidently be equal—that is,
$$B = \frac{dA}{dx},\ C = \frac{dB}{2dx} \text{ and } D = \frac{dC}{3dx}, \qquad (6)$$
the rates of which are, regarding dx as constant,
$$dB = \frac{d^2A}{dx},\quad dC = \frac{d^2B}{2dx},\quad dD = \frac{d^2C}{3dx}; \qquad (7)$$
also $$d^2B = \frac{d^3A}{dx},\quad d^2C = \frac{d^3B}{2dx}. \qquad (7)$$

From (6) and (7) it will be readily found that
$$B = \frac{dA}{dx},\quad C = \frac{d^2A}{2dx^2} \text{ and } D = \frac{d^2B}{2\cdot 3dx^2} = \frac{d^3A}{2\cdot 3dx^3}.$$

Substituting these values of B, C, and D in (2) will give
$$u = A + \frac{dA}{dx}y + \frac{d^2A}{2dx^2}y^2 + \frac{d^3A}{2\cdot 3dx^3}y^3 + \text{etc.}, \qquad (8)$$
known as the theorem of Taylor.

In like manner the development of $u = f(x - y)$ will be found to be

$$u = A - \frac{dA}{dx}y + \frac{d^2A}{2dx^2}y^2 - \frac{d^3A}{2 \cdot 3dx^3}y^3 + \text{etc.} \qquad (9)$$

Although this theorem gives the general development of every function of the sum or difference of two variables correctly, yet in some particular cases a certain value may be ascribed to the variable x which will render the development impossible, as will be indicated by some of the ratal coefficients of the development becoming equal to infinity: thus, if in

$$u = a + (b + x - y)^{1/2}$$

y be made equal to zero, then

$$A = a + (b + x)^{1/2},$$

the first and second ratal coefficients of which are

$$\frac{dA}{dx} = \frac{1}{2(b + x)^{1/2}} \text{ and } \frac{d^2A}{dx^2} = -\frac{1}{4(b + x)^{3/2}},$$

both of which become equal to infinity when $x = -b$.

For an exemplification of the theorem, develop

$$u = (x + y)^n.$$

Making $y = 0$ gives

$$u = A = x^n,$$

the successive ratal coefficients of which are, from Art. 20,

$$\frac{dA}{dx} = nx^{n-1}, \quad \frac{d^2A}{dx^2} = n(n-1)x^{n-2},$$

$$\frac{d^3A}{dx^3} = n(n-1)(n-2)x^{n-3}, \text{ etc.}$$

Substituting these values in formula (8), the result is

$$u = x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2 +$$

$$\frac{n(n-1)(n-2)}{2 \cdot 3}x^{n-3}y^3 + \text{etc.},$$

the same as found by the binomial theorem.

EXAMPLES

1. $u = (x + y)^{1/2}$ 2. $u = (x + ay)^{-2}$

TRANSCENDENTAL FUNCTIONS

26. Let the function be

$$u = a^x. \tag{1}$$

Assuming $a = 1 + c,$

then $u = (1 + c)^x,$

the development of which, by the binomial theorem, is

$$u = 1 + xc + \frac{x(x-1)}{2}c^2 + \frac{x(x-1)(x-2)}{2 \cdot 3}c^3 + \frac{x(x-1)(x-2)(x-3)}{2 \cdot 3 \cdot 4}c^4 + \text{etc.} \tag{2}$$

Passing to the rate,

$$du = (c + \frac{2x-1}{2}c^2 + \frac{3x^2 - 6x + 2}{2 \cdot 3}c^3 + \frac{4x^3 - 18x^2 + 22x - 6}{2 \cdot 3 \cdot 4}c^4 + \text{etc.})\, dx. \tag{3}$$

Dividing each member of (3) by the corresponding members of (2) gives

$$\frac{du}{u} = (c - \frac{c^2}{2} + \frac{c^3}{3} - \frac{c^4}{4} + \text{etc.})\, dx, \tag{4}$$

which is the ordinary logarithmic series for $\log(1 + c)$, that is

$$\log(1 + c) = c - \frac{c^2}{2} + \frac{c^3}{3} - \frac{c^4}{4} + \text{etc.} \tag{5}$$

But $1 + c = a$, therefore

$$\frac{du}{u} = dx \log a.$$

Substituting for u its value and multiplying by a^x,

$$du = a^x dx \log a. \tag{6}$$

Substituting in (4) for u and c their values and multiplying by a^x, the result is

$$du = a^x (\frac{a-1}{1} - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} + \text{etc.})\, dx.$$

Assuming

$$\frac{a-1}{1}-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}-\frac{(a-1)^4}{4}+\text{etc.}=e$$

then $$du=a^x\,edx, \qquad (7)$$

wherein e is dependent on a for its value, which may be determined by Maclaurin's theorem.

Since the rate of $u=a^x$ is $du=a^xedx$, (7), it is evident, e being constant, that the rate of $du=a^xedx$ *is* $d^2u=a^xe^2dx^2$ and the rate of $d^2u=a^xe^2dx^2$ is $d^3u=a^xe^3dx^3$. Therefore, the ratal coefficients are, from Art. 20,

$$\frac{du}{dx}=a^xe,\quad \frac{d^2u}{dx^2}=a^xe^2,\quad \frac{d^3u}{dx^3}=a^xe^3,\ \text{etc.}$$

Making $x=0$ in (1), also in the ratal coefficients, it will be found that

$$u=1,\quad \frac{du}{dx}=e,\quad \frac{d^2u}{dx^2}=e^2,\quad \frac{d^3u}{dx^3}=e^3,\ \text{etc.}$$

Substituting these values in (7), Art. 24, gives

$$u=a^x=1+\frac{ex}{1}+\frac{e^2x^2}{2}+\frac{e^3x^3}{2\cdot 3}+\text{etc.}$$

If $x=\frac{1}{e}$, then

$$a^x=1+\frac{1}{1}+\frac{1}{2}+\frac{1}{2\cdot 3}+\text{etc.}$$

The sum of the first twelve terms of this series is 2.7182818, which is the base of the Naperian system of logarithms; hence e is the Naperian logarithm of a. Therefore, substituting $\log a$ for e in (7), then

$$du=a^xdx\log a,$$

the same as (6).

Hence *the rate of an exponential function is the function into the rate of the exponent multiplied by the Naperian logarithm of the constant of which the variable is the exponent.*

27. Resuming (1) of the last article, transposing it, and taking the logarithm of both members give

$$x \log a = \log u,$$

whence $$x = \frac{\log u}{\log a}. \qquad (1)$$

Now it has been shown in the last article, that the rate of $u = a^x$ is

$$du = a^x dx \log a,$$

whence $$dx = \frac{du}{a^x \log a}$$

or, substituting for a^x its value, from (1) of Art. 26,

$$dx = \frac{du}{u \log a}, \qquad (2)$$

the rate of (1).

If a is the base of a system of logarithms, then x is the logarithm of u in that system, and $\frac{1}{\log a}$ is the modulus of the system; therefore representing $\frac{1}{\log a}$ by M, (2) becomes

$$dx = M \frac{du}{u}. \qquad (3)$$

Hence *the rate of the logarithm of a quantity is the modulus of the system into the rate of the quantity, divided by the quantity itself.*

The modulus of the Naperian system of logarithms is unity; therefore if the logarithms are taken in the Naperian system (3) becomes

$$dx = \frac{du}{u}.$$

Hence *the rate of the Naperian logarithm of a quantity is the rate of the quantity divided by the quantity itself.*

28. To determine the rate of

$$u = v^x,$$

in which both v and x are variables.

Taking the logarithm of both members,

$$\log u = x \log v,$$

and passing to the rate, by Arts. 26 and 9,

$$\frac{du}{u} = \frac{xdv}{v} + dx \log v,$$

or

$$du = \frac{uxdv}{v} + udx \log v.$$

Substituting v^x for u and reducing,

$$du = xv^{x-1}dv + v^x dx \log v. \qquad (1)$$

Hence *the rate of a variable quantity having a variable exponent is the sum of the rates obtained, first under the supposition that the quantity varies and the exponent remains constant, then that the exponent varies and the quantity remains constant.*

29. Take $$u = \log (1 + x), \qquad (1)$$
in which u is the Naperian logarithm of $1 + x$, and the successive ratal coefficients are

$$\frac{du}{dx} = \frac{1}{1+x}, \quad \frac{d^2u}{dx^2} = -\frac{1}{(1+x)^2}, \quad \frac{d^3u}{dx^3} = \frac{1 \cdot 2}{(1+x)^3}, \text{ etc.}$$

Making $x = 0$ in (1), also in the ratal coefficients, then

$$u = 0, \quad \frac{du}{dx} = 1, \quad \frac{d^2u}{dx^2} = -1,$$

$$\frac{d^3u}{dx^3} = 1 \cdot 2, \quad \frac{d^4u}{dx^4} = -1 \cdot 2 \cdot 3, \text{ etc.}$$

and consequently, by substitution in (7) of Art. 24,

$$u = \log (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \text{etc.} \qquad (2)$$

[see (5) of Art. 26].

Developing $$u = \log (1 - x)$$
in like manner, it will be found that

$$u = \log (1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \text{etc.} \qquad (3)$$

Subtracting (3) from (2) gives

$$\log(1+x) - \log(1-x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \text{etc.}\right) \quad (4)$$

Assuming $$\frac{1+x}{1-x} = \frac{v+1}{v},$$

then $$x = \frac{1}{2v+1};$$

therefore, since

$$\log\left(\frac{v+1}{v}\right) = \log(v+1) - \log v,$$

by substituting the value of x in (4), the following is obtained:

$$\log(v+1) - \log v = 2\left[\frac{1}{(2v+1)} + \frac{1}{3(2v+1)^3} + \frac{1}{5(2v+1)^5} + \text{etc.}\right],$$

or

$$\log(v+1) = \log v + 2\left[\frac{1}{(2v+1)} + \frac{1}{3(2v+1)^3} + \frac{1}{5(2v+1)^5} + \text{etc.}\right],$$

by which the logarithm of $v+1$ can readily be found when the logarithm of v is known. Thus, if

$$v=1,\ \log 2 = 0 + 2\left(\frac{1}{3} + \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} + \text{etc.}\right) = 0.69314718$$

$$v=2,\ \log 3 = \log 2 + 2\left(\frac{1}{5} + \frac{1}{3\cdot 5^3} + \frac{1}{5\cdot 5^5} + \text{etc.}\right) = 1.09861229$$

$$v=3,\ \log 4 = \log 2 + \log 2 = 1.38629436$$

$$v=4,\ \log 5 = \log 4 + 2\left(\frac{1}{9} + \frac{1}{3\cdot 9^3} + \frac{1}{5\cdot 9^5} + \text{etc.}\right) = 1.60943791$$

$$v=5,\ \log 6 = \log 2 + \log 3 = 1.79175947$$

$$v=6,\ \log 7 = \log 6 + 2\left(\frac{1}{13} + \frac{1}{3\cdot 13^3} + \frac{1}{5\cdot 13^5} + \text{etc.}\right) = 1.94591014$$

$$v=7,\ \log 8 = \log 2 + \log 4 = 2.07944154$$

$$v=8,\ \log 9 = \log 3 + \log 3 = 2.19722458$$

$$v=9,\ \log 10 = \log 2 + \log 5 = 2.30258509$$

The logarithm of 10 in the common system is 1, and in the Naperian system is 2.30258509; 1 divided by 2.30258509 is 0.43429448, the modulus of the common system, usually designated by M.

To avoid inconvenience, the Naperian logarithms are generally used in this work. Whenever the common system may desired, it will be necessary to multiply by the modulus of that system.

EXAMPLES

To determine the rate of

$$u = \log \frac{x}{(a^2 + x^2)^{1/2}}.$$

Assuming $$v = \frac{x}{(a^2 + x^2)^{1/2}}, \qquad (1)$$

then $$u = \log v \text{ and } du = \frac{dv}{v}. \qquad (2)$$

From (1), by passing to the rate,

$$dv = \frac{(a^2 + x^2)^{1/2}\,dx - x^2\,(a^2 + x^2)^{-1/2}\,dx}{(a^2 + x^2)},$$

or, reducing, $$dv = \frac{a^2 dx}{(a^2 + x^2)^{3/2}}. \qquad (3)$$

Substituting for v and dv their values in (2) gives

$$du = \frac{a^2 dx}{x\,(a^2 + x^2)}.$$

1. $u = \dfrac{a^2 + b^x}{a^x + b^2}$

2. $u = \dfrac{\log x}{\log y}$

3. $u = \log \dfrac{(1 + x) + (1 + y)}{(1 + x) - (1 + y)}$

What is the rate of the common *logarithm* 4300?

Illustrations of Principles Relative to Curves

30. If a particle impelled from A toward B along the curve

APB (see figure) be left to itself at any point in the curve,

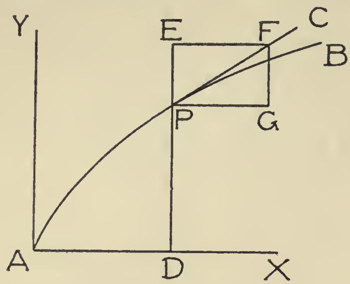


Fig. 5

as at P, it is obvious that it would then proceed at a uniform rate toward C along the straight line PC tangent to the curve at P.

For the point P, let x represent the abscissa AD, y the ordinate DP, and z the curve AP. Extend DP to E, and draw EF and PG parallel to AX; also draw EP and FG parallel to AY. Then dx will be represented by PG, dy by EP, and dz by PF; hence

$$dz^2 = dx^2 + dy^2 \text{ or } dz = (dx^2 + dy^2)^{\frac{1}{2}}.$$

Circular Functions

31. To determine the rate of

$$u = \sin x,$$

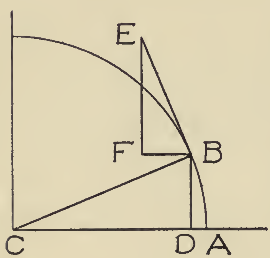


Fig. 6

let the radius $AC = BC = R$ (see figure), the arc $AB = x$, and, as the case may be, let u represent the sine, cosine, tangent, etc. of the arc x; then we have, by Art. 30, $BE = dx$, BD equal to the sine, CD the cosine, EF the rate of the sine, and BF the rate of the cosine; hence

$$BC : CD :: BE : EF$$

$$\text{or } R : \cos x :: dx : du,$$

whence
$$du = \frac{\cos x dx}{R}.$$

Therefore *the rate of the sine of an arc is equal to the cosine of the arc into the rate of the arc, divided by the radius.*

If
$$u = \cos x,$$

then, since u, the cosine CD, is a decreasing function of the arc x, its rate is negative; therefore

$$R : \sin x :: dx : -du \text{ or } du = -\frac{\sin x dx}{R}.$$

Hence *the rate of the cosine is minus the sine into the rate of the arc, divided by the radius.*

If $$u = \text{vers } x = R - \cos x,$$

then $$du = \frac{\sin x dx}{R}.$$

Hence *the rate of the versed sine is the sine into the rate of the arc, divided by the radius.*

If $$u = \tan x = \frac{R \sin x}{\cos x},$$

then, by passing to the rate, by Art. 19,

$$du = \frac{\cos^2 x dx + \sin^2 x dx}{\cos^2 x},$$

or, since $\cos^2 x + \sin^2 x = R^2$,

$$du = \frac{R^2 dx}{\cos^2 x}.$$

Therefore *the rate of the tangent of an arc is equal to the square of the radius into the rate of the arc, divided by the square of the cosine.*

If $$u = \cot x = \frac{R \cos x}{\sin x},$$

then, passing to the rate, by Art. 19,

$$du = \frac{-(\cos^2 x + \sin^2 x)\, dx}{\sin^2 x},$$

or, since $\cos^2 x + \sin^2 x = R^2$,

$$du = -\frac{R^2 dx}{\sin^2 x}.$$

Therefore *the rate of the cotangent is equal to minus the square of the radius into the rate of the arc, divided by the square of the sine.*

If $$u = \sec x = \frac{R^2}{\cos x},$$

then, passing to the rate,

$$du = \frac{R^2 \sin x dx}{\cos^2 x}.$$

Therefore *the rate of the secant is the square of the radius into the sine multiplied by the rate of the arc, divided by the square of the cosine.*

If $$u = \text{cosec}\, x = \frac{R^2}{\sin x},$$

then $$du = -\frac{R^2 \cos x dx}{\sin^2 x}.$$

Therefore *the rate of the cosecant is equal to minus the square of the radius into the cosine multiplied by the rate of the arc, divided by the square of the sine.*

If $R = 1$, then for

$u = \sin x,$ $\quad du = \cos x dx,$

$u = \cos x,$ $\quad du = -\sin x dx,$

$u = \text{vers}\, x,$ $\quad du = \sin x dx,$

$u = \tan x,$ $\quad du = \dfrac{dx}{\cos^2 x}$, etc.

If $$u = \sin rx, \tag{1}$$

by assuming $rx = v$, then will $\sin rx = \sin v$,

whence $dv = rdx$ and $du = d(\sin rx) = \cos v dv.$ (2)

Substituting these values of v and dv in (1), then

$$du = r \cos rx dx.$$

If $$u = \sin^n x, \tag{3}$$

by assuming $\sin x = v$, then will $\sin^n x = v^n$,

whence $dv = \cos x dx$ and $du = d(\sin^n x) = nv^{n-1}dv.$ (4)

Substituting these values of v and dv in (4) gives

$$du = n \sin^{n-1} x \cos x dx.$$

If $$u = \sin^n x \sin rx, \tag{5}$$

then $du = d(\sin^n x) \sin rx + d(\sin rx) \sin^n x;$

but, as shown, $d(\sin^n x) = n \sin^{n-1} x \cos x dx$

and $d(\sin rx) = r \cos rx dx;$

therefore

$$du = (n \sin^{n-1} x \cos x \sin rx + r \cos rx \sin^n x)\, dx,$$

or $$du = \sin^{n-1} x\, (n \cos x \sin rx + r \cos rx \sin x)\, dx.$$

By use of these equations, the rates of like expressions of the cosine, versed sine, tangent, etc., can be determined.

32. For radius R and

$$u = \log \sin x,$$

by passing to the rate, by Arts. 27 and 31,

$$u = \frac{\cos x dx}{R \sin x},$$

the rate of the logarithmic sine of the arc x.

If $$u = \log \cos x,$$

then $$du = -\frac{\sin x dx}{R \cos x},$$

the rate of the logarithmic cosine of the arc x.

If $$u = \log \text{vers } x$$

or its equivalent, $u = \log (R - \cos x)$,

then $$du = \frac{\sin x dx}{R (R - \cos x)}$$

or $$du = \frac{\sin x dx}{R \text{ vers } x},$$

the rate of the logarithmic versed sine of the arc x.

If $$u = \log \tan x$$

or, since $\tan x = \dfrac{R \sin x}{\cos x}$,

$$u = \log (\frac{R \sin x}{\cos x});$$

then, passing to the rate, by Arts. 27 and 31, and reducing,

$$du = \frac{\cos^2 x dx + \sin^2 x dx}{R \sin x \cos x}$$

or, since $\cos^2 x + \sin^2 x = R^2$,

$$du = \frac{R dx}{\sin x \cos x},$$

the rate of the logarithmic tangent of the arc x.

If $$u = \log \cot x = \log \frac{R \cos x}{\sin x};$$

then $$du = \frac{-(\sin^2 x dx + \cos^2 x dx)}{\sin x^2} \div \frac{R \cos x}{\sin x}$$

or, since $\sin^2 x + \cos^2 x = R^2$, by reducing

$$du = -\frac{R dx}{\sin x \cos x},$$

the rate of the logarithmic cotangent of the arc x.

If $$u = \log \sec x = \log \frac{R^2}{\cos x},$$

then $$du = \frac{R^2 \sin x dx}{R \cos^2 x} \div \frac{R^2}{\cos x} = \frac{\sin x dx}{R \cos x}$$

or, since $\frac{R \sin x}{\cos x} = \tan x$,

$$du = \frac{\tan x dx}{R^2},$$

the rate of the logarithmic secant of the arc x.

In like manner the rate of

$$u = \log \operatorname{cosec} x$$

will be found to be $$du = -\frac{\cot x dx}{R^2}.$$

In using the common system of logarithms, for

$u = \log \sin x$, $\quad du = \frac{M \cos x dx}{R \sin x}$;

$u = \log \operatorname{vers} x$, $\quad du = \frac{M \sin x dx}{R \operatorname{vers} x}$;

$u = \log \cos x$, $\quad du = -\frac{M \sin x dx}{R \cos x}$;

$u = \log \tan x$, $\quad du = \frac{M R dx}{\sin x \cos x}$;

$u = \log \cot x$, $\quad du = -\frac{M R dx}{\sin x \cos x}$, etc.

33. It is often desirable to have the rate of the arc in terms of that of its sine, cosine, tangent, etc., and for this

purpose the following expressions are employed: namely, $x = \sin^{-1} u$, $x = \cos^{-1} u$, $x = \tan^{-1} u$, etc., x being the arc and u its sine, cosine, etc.

Let $$x = \sin^{-1} u,$$
its equivalent being $$u = \sin x,$$
the rate of which is, by Art. 31,

$$du = \frac{\cos x dx}{R},$$

whence $$dx = \frac{R du}{\cos x},$$

or, since $\sin x = u$, consequently $\cos x = (R^2 - u^2)^{1/2}$,

$$dx = \frac{R du}{(R^2 - u^2)^{1/2}},$$

the rate of the arc in terms of the sine and its rate.

In like manner, if $$x = \cos^{-1} u,$$
it will be found that $$dx = -\frac{R du}{(R^2 - u^2)^{1/2}},$$

the rate of the arc in terms of the cosine and its rate.

If $$x = \text{vers}^{-1} u,$$
its equivalent being $$u = \text{vers}\, x = R - \cos x,$$
the rate of which is, by Art. 31,

$$du = \frac{\sin x dx}{R}$$

whence $$dx = \frac{R du}{\sin x}.$$

Now $\sin x = (R^2 - \cos x^2)^{1/2}$,
but

$$\cos x = R - \text{vers}\, x = R - u;$$

therefore

$$\sin x = \{R^2 - (R - u)^2\}^{1/2} = (2Ru - u^2)^{1/2};$$

hence $$dx = \frac{R du}{(2Ru - u^2)^{1/2}},$$

the rate of the arc in terms of the versed sine and its rate.

If $x = \tan^{-1} u,$

the equivalent being

$$x = \tan x,$$

the rate of which is, by Art. 31,

$$du = \frac{R^2 dx}{\cos^2 x}$$

whence $$dx = \frac{\cos^2 x du}{R^2}. \quad (1)$$

But $\sec^2 x : R^2 :: R^2 : \cos^2 x$, or, since $\sec^2 x = R^2 + u^2$,

$$R^2 + u^2 : R^2 :: R^2 : \cos^2 x,$$

whence $$\cos^2 x = \frac{R^4}{R^2 + u^2}.$$

Substituting this value of $\cos^2 x$ in (1) gives

$$dx = \frac{R^2 du}{R^2 + u^2},$$

the rate of the arc in terms of the tangent and its rate.

In like manner, if $x = \cot^{-1} u,$

it will be found that $dx = -\dfrac{R^2\, du}{R^2 + u^2},$

the rate of the arc in terms of the cotangent and its rate.

34. By means of Maclaurin's theorem, $\sin x$, $\cos x$, etc. can be developed in terms of x: thus, if $R = 1$, the ratal coefficients of $u = \sin x$ are

$$\frac{du}{dx} = \cos x \qquad \frac{d^2u}{dx^2} = -\sin x$$

$$\frac{d^3u}{dx^3} = -\cos x \qquad \frac{d^4u}{dx^4} = \sin x$$

$$\frac{d^5u}{dx^5} = \cos x, \text{ etc.}$$

Making $x = 0$, then, from Art. 24,

$$A = 0, \ \frac{du}{dx} = 1, \ \frac{d^2u}{dx^2} = 0, \ \frac{d^3u}{dx^3} = -1,$$

$$\frac{d^4u}{dx^4}=0,\ \frac{d^5u}{dx^5}=1, \text{ etc.}$$

By substituting these values in Maclaurin's theorem the following is obtained:

$$u=\sin x=\frac{x}{1}-\frac{x^3}{2\cdot 3}+\frac{x^5}{2\cdot 3\cdot 4\cdot 5}, - \text{etc.}$$

Proceeding in like manner, it will be found from $u=\cos x$

that $$u=1-\frac{x^2}{1}+\frac{x^4}{2\cdot 3\cdot 4} - \text{etc.}$$

EXAMPLES

Determine the rates of the following:

$u=\sin x^2$

$u=\tan x^2$

$u=\sin x \cos x$

$u=\cos x \sin x$

$u=\tan x \log \cot x$

$u=\log \tan x+\log \cot x$

$x=\sin^{-1} 2u\,(1-u)$

$x=\cos^{-1}\dfrac{u}{a-u}$

$x=\tan^{-1}\dfrac{b}{u}$

Develop $x=\sin^{-1} u.$

Vanishing Fractions

35. A vanishing fraction is one which reduces to the form $\frac{0}{0}$ when a particular value is given to the variable. Thus,

$$\frac{b\,(x^2-a^2)}{c\,(x-a)}$$

reduces to $\frac{0}{0}$ when $x=a$. This, it will be seen, is owing to a factor common to both numerator and denominator which reduces to zero for $x=a$.

Let the equation be

$$u=\frac{(x^m-a^m)^r}{(x^n-a^n)^s} \qquad (1)$$

and put it under the form

$$u = \frac{(x-a)^r \{x^{m-1} + amx^{m-2} + a^2 m (m-1) x^{m-3} + \dots a^{m-1}\}^r}{(x-a)^s \{x^{n-1} + anx^{n-2} + a^2 n (n-1) x^{n-3} + \dots a^{n-1}\}^s}$$

Now the right-hand portion of this fraction does not reduce to zero for $x = a$; therefore, making it equal to R, then

$$u = R \frac{(x-a)^r}{(x-a)^s}. \tag{2}$$

Let $(x-a)^r = P$, and $(x-a)^s = Q$, both of which reduce to zero for $x = a$; then

$$u = \frac{RP}{Q}, \tag{3}$$

or

$$Qu = RP.$$

Passing to the rate,

$$Qdu + udQ = RdP + PdR,$$

but $Q = 0$ and $P = 0$; therefore

$$udQ = RdP$$

or

$$u = R \frac{dP}{dQ};$$

hence, since $u = \frac{RP}{Q}$, (3)

$$u = R \frac{P}{Q} = R \frac{dP}{dQ}.$$

If both dP and dQ reduce to zero for $x = a$, then by passing to the rate again

$$R \frac{dP}{dQ} = R \frac{d^2P}{d^2Q};$$

therefore

$$u = R \frac{P}{Q} = R \frac{dP}{dQ} = R \frac{d^2P}{d^2Q}.$$

Should this also reduce to zero for $x = a$, by continuing the process a fraction may be found which will not reduce to zero for $x = a$, and thus the true value of the primitive fraction will be obtained.*

* It will be observed that the process employed simply eliminates the vanishing factor, thus giving the real value of the fraction.

Let $$u = (bx^2 + cx)\left(\frac{x^2 - a^2}{x - a}\right), \tag{4}$$

in which $m = 2$, $n = 1$, and $bx^2 + cx$ represents R, and its rate is

$$u = (bx^2 + cx)\frac{d(x^2 - a^2)}{d(x - a)} = (bx^2 + cx)\frac{2xdx}{dx} = 2bx^3 + 2cx^2$$

or, when $x = a$, $u = 2a^3b + 2a^2c$. (5)

Multiplying $(bx^2 + cx)$ by $(x^2 - a^2)$ in (4) gives

$$u = \frac{bx^4 + cx^3 - a^2bx^2 - a^2cx}{x - a}.$$

Therefore

$$u = \frac{(4bx^3 + 3cx^2 - 2a^2bx - a^2c)\,dx}{dx} =$$

$$4bx^3 + 3cx^2 - 2a^2bx - a^2c,$$

or, when $x = a$, $$u = 2a^3b + 2a^2c,$$

the same result as (5).

If $$u = \frac{x^n - a^n}{x - a},$$

then $$u = \frac{d(x^n - a^n)}{d(x - a)} = \frac{nx^{n-1}dx}{dx} = nx^{n-1}$$

or, when $x = a$, $$u = na^{n-1}.$$

If $$u = \frac{(x^2 - a^2)^{3/2}}{(x - a)^{3/2}},$$

by squaring and then taking the cube root,

$$u^{2/3} = \frac{(x^2 - a^2)}{(x - a)}.$$

Therefore $$u^{2/3} = \frac{2xdx}{dx} = 2x$$

or, when $x = a$,

$$u^{2/3} = 2a \text{ or } u = (2a)^{3/2}.$$

EXAMPLES

$$u=\frac{(x^2-a^2)^3}{(x^3-a^3)^2} \qquad u=\frac{a^3x^3\,(a-x)^{3/2}}{(a^2-x^2)^{2/3}}$$

$$u=\frac{(x^3-2ax^2+a^3)^{1/2}}{(x^2-a^2)^{1/2}} \qquad u=\frac{(2ax^2-2a^2)^{1/2}}{(x^3+a^2x-2a^3)^{1/2}}$$

$$u=\frac{x^3-2ax^2+a^2x}{x^2-a^2} \qquad u=\frac{a-x+a\log x-a\log a}{a-(2ax-x)^{1/2}}$$

Determine the value of $u=\dfrac{a^x-c^x}{x\,(x+c)}$, when $x=0$.

Determine the value of

$$u=\frac{\cos x-\sin x+1}{\cos x+\sin x-1}, \text{ when } x=90^\circ.$$

In some cases both P and Q become infinite for a particular value of the variable: thus

$$u=\frac{\tan x}{\cot 2x}$$

becomes $\dfrac{\infty}{\infty}$ when $x=90^\circ$.

Now $\quad \tan x=\dfrac{1}{\cot x}$ and $\cot 2x=\dfrac{1}{\tan 2x}$,

therefore $\dfrac{\tan x}{\cot 2x}=\dfrac{\tan 2x}{\cot x}=\dfrac{0}{0}$, when $x=90^\circ$.

Hence

$$u=\frac{\dfrac{2dx}{\cos^2 2x}}{\dfrac{-dx}{\sin^2 x}}=-\frac{2\sin^2 x}{\cos^2 2x},$$

or, since $\sin x=1$ and $\cos 2x=-1$ when $x=90^\circ$,

$$u=-2.$$

Sometimes in a product one factor becomes zero and the other infinite for a particular value of the variable: thus, in

$$u=(1-x)\tan\frac{1}{2}\pi x \tag{6}$$

$(1-x)=0$ and $\tan\frac{1}{2}\pi x=\infty$, when $x=1$ and $\frac{1}{2}\pi=90^\circ$.

Since $\tan\frac{1}{2}\pi x=\dfrac{1}{\cot\frac{1}{2}\pi x}$, (6) can be written thus:

$$u=\frac{1-x}{\cot\frac{1}{2}\pi x},$$

then
$$u=\frac{-dx}{\dfrac{-\frac{1}{2}\pi\,dx}{\sin^2\frac{1}{2}\pi x}}=\frac{2\sin^2\frac{1}{2}\pi x}{\pi}.$$

Therefore, when $x=1$ and $\frac{1}{2}\pi=90^\circ$, since $\sin\frac{1}{2}\pi x$ then equals 1,

$$u=\frac{2}{\pi}.$$

Of the difference of two quantities, both sometimes become infinite for a particular value of the variable: thus, in

$$u=\frac{x}{x-1}-\frac{1}{\log x}, \qquad (7)$$

when $x=1$, both $\dfrac{x}{x-1}$ and $\dfrac{1}{\log x}$ become infinite.

In this case put (7) under the form

$$u=\frac{x\log x-x+1}{(x-1)\log x}$$

which becomes $\frac{0}{0}$ when $x=1$; therefore

$$u=\frac{d\,(x\log x-x+1)}{d\,(x-1)\log x}=\frac{x\log x}{x\log x+x-1}.$$

This also becomes $\frac{0}{0}$ when $x=1$, consequently

$$u=\frac{dx\log x+\frac{xdx}{x}}{dx\log x+\frac{xdx}{x}+dx}=\frac{\log x+1}{\log x+1+1}$$

or, when $x=1$, since $\log 1=0$,

$$u=\frac{1}{2}.$$

EXAMPLE

Determine the value of

$$u=x\tan x-\frac{1}{2\cos x}, \text{ when } x=90^\circ.$$

Curves Referred to Rectangular Coördinates

36. Signification of the first and second ratal coefficients.

Every curve or line referred to rectangular coördinates may generally be represented by the equation

$$y=f(x)$$

in which x represents any abscissa, as AB, and y the corresponding ordinate BP, of the curve CPD (see figure).

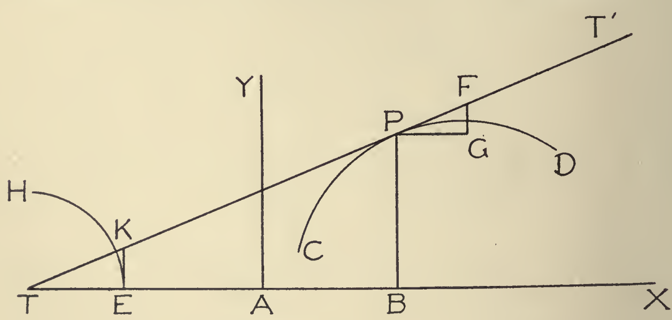


Fig. 7

Draw TT' tangent to the curve CPD at P, and with radius unity draw the arc EH, also draw EK tangent to EH. Then the angle $T'TX$ will be the angle of tangency, so called, and

EK its tangent, which is designated by t. Now let dx be represented by PG, and dy by GF (see Art. 30); then will

$$dx : dy :: TE : EK$$

or, since $TE = 1$, and $EK = t$, by representing the rate of $f(x)$ by $f'(x)$

$$dx : dy :: 1 : t,$$

whence $$t = \frac{dy}{dx} = f'(x).$$

Passing to the rate and representing the rate of $f'(x)$ by $f''(x)$,

$$\frac{dt}{dx} = \frac{d^2y}{dx^2} = f''(x).$$

Hence *the first ratal coefficient of the equation of a curve represents the tangent of the angle of tangency of any point of the curve and the second ratal coefficient represents the rate of variation of the tangent of the angle of tangency.*

37. Of a curve concave to the axis of X (Fig. 8), the angle of tangency and consequently its tangent t decrease as the ordinate increases, as is clearly shown by the tangents TP and $T'P'$ of the curve $CPP'D$.

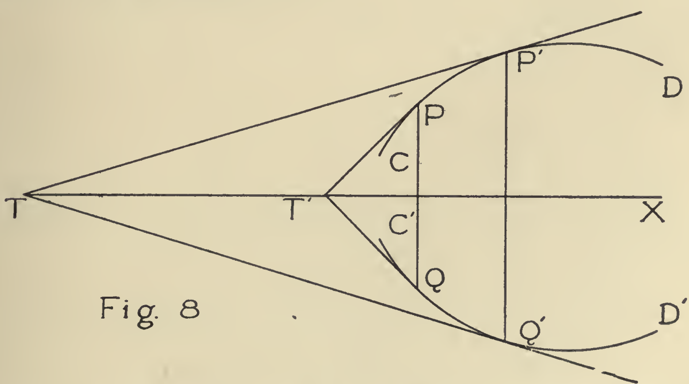


Fig. 8

This, it will be observed, is also true of the curve $C'QQ'D'$, as indicated by the tangents TQ and $T'Q'$; but, lying below the axis of X, the angle of tangency and its tangent, as well as the ordinate, are negative, while those above the axis are positive.

Now, since the rate of a positive decreasing function is negative and that of a negative decreasing function is positive, the rate of the tangent of the angle of tangency of the curve $CP'D$ is negative, while that of the curve $C'QQ'D'$ is positive. Therefore, since t represents the tangent of the angle of tangency, *when a curve is concave to the axis of x and its ordinate is positive, dt, and consequently the second ratal coefficient of the equation of the curve, are negative, but positive when the ordinate is negative.*

If the curve is convex to the axis of X (Fig. 9), the angle of tangency, and consequently its tangent t, increase as the ordinate increases, as is shown by the tangents TP and $T'P'$ of the curve $CPP'D$.

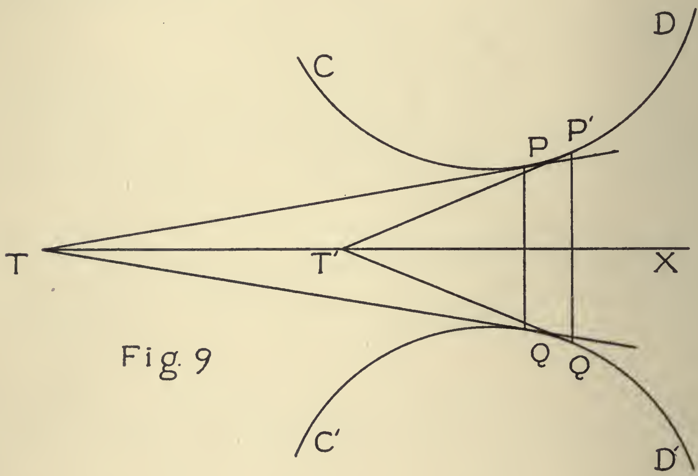


Fig. 9

This is also true of the curve $C'QQ'D'$, as indicated by the tangents TQ and $T'Q'$; but as they lie below the axis of X, the angle of tangency and its tangent, as well as the ordinate, are negative.

Therefore, since the rate of a positive increasing function is positive, and that of a negative increasing function is negative, the rate of the tangent of the angle of tangency for any point of the curve $CPP'D$ is positive, while that of the tangency for any point of the curve $C'QQ'D'$ is negative.

Hence *when a curve is convex to the axis of X, and its ordinate is positive and increasing, dt, and consequently the second ratal coefficient of the equation of the curve, are positive, but negative when the ordinate is negative.*

From what precedes, the following conclusion is evident:

When the second ratal coefficient of the equation of a curve is negative, the curve is either concave to and above the axis of X or convex to and below it; but when the second ratal coefficient is positive, the curve is either concave to and below the axis or convex to and above it.

Sometimes a particular value of x, as $x=a$, will make $\frac{d^2y}{dx^2}=0$. In this case, substitute $a \pm v$ for x; then, for a small value of v, if $\frac{d^2y}{dx^2}$ and the ordinate corresponding to $x=a$ have contrary signs, the curve is concave to the axis of X at a point in the curve whose abscissa is $x=a$, but if like signs, convex.

E. g., let the equation of the curve be

$$y = x^5 - 5x^4 + 40x^2 - 80x + 58.$$

Passing to the rate twice,

$$\frac{d^2y}{dx^2} = 20x^3 - 60x^2 + 80, \tag{1}$$

in which $\frac{d^2y}{dx^2}=0$ when $x=2$; therefore, substituting $2 \pm v$ for x in (1) gives

$$\frac{d^2y}{dx^2} = 60v^2 \pm 20v^3$$

or

$$\frac{d^2y}{dx^2} = 20v^2\,(3 \pm v),$$

which is positive for any value of $v < 3$. Hence, since $y=10$ when $x=2$, the curve is convex to the axis of X at the point whose abscissa is $x=2$.

Determine whether the curve whose equation is

$$y = 5 + 4x - x^2,$$

is concave or convex to the axis of X.

Ratal Equations of Lines

38. A ratal equation of a line is one which shows the relation between the coördinates and their rates, and, being independent of the values of the constants which enter the primitive

equation, determines the general nature of the line without regard to its magnitude.

First take the general equation of lines of the first order

$$y = ax + b,$$

whence

$$\frac{dy}{dx} = a,$$

a result which is the same for all values of b. This equation represents the tangent of the angle of tangency (see Art. 36) and is the *first ratal equation of lines of the first order.*

Passing to the rate again

$$d\frac{dy}{dx} = 0 \text{ or } \frac{d^2y}{dx^2} = 0,$$

an equation entirely independent of the values of a and b and consequently equally applicable to every line of the first order which can be drawn in the plane of the coördinate axes. It is called the *general ratal equation of lines of the first order.*

The equation $\frac{d^2y}{dx^2} = 0$ shows that the tangent of the angle of tangency has no variation (see Art. 37); hence every line of the first order must necessarily be a straight line.

39. In the general equation of lines of the second order

$$y^2 = ax^2 + bx + c, \tag{1}$$

passing to the rate thrice will give

$$\frac{2ydy}{dx} = 2ax + b \tag{2}$$

$$\frac{dy^2}{dx^2} + \frac{yd^2y}{dx^2} = a \tag{3}$$

$$\frac{dyd^2y}{dx^3} + \frac{yd^3y}{dx^3} = 0; \tag{4}$$

equations (2), (3), and (4) are respectively the first, second, and general ratal equations of lines of the second order.

When the origin of coördinates is at the vertex of the transverse axis, the general equation of lines of the second order is

$$y^2 = ax^2 + bx. \tag{5}$$

Passing to the rate twice

$$2ydy = 2axdx + bdx \qquad (6)$$

and

$$dy^2 + yd^2y = adx^2. \qquad (7)$$

Eliminating a and b in (5) by means of (6) and (7)

$$y^2dx^2 + x^2dy^2 + x^2yd^2y - 2xydxdy = 0,$$

a general ratal equation of lines of the second order, when the origin of the coördinates is at the vertex of the transverse axis, differing from (4) on account of passing to the rate but twice.

Determine the general ratal equations of the circle, parabola, and ellipse.

Tangents and Normals

40. If x and y represent the coördinates of any point of the curve APC, and z the corresponding arc AP, Art. 30, dx will be represented by DP, dy by DE, and dz by PE, a tangent to the curve at the point P; also TR represents the subtangent, TP the tangent, RN the subnormal, and PN the normal, each of which are obtained as follows:

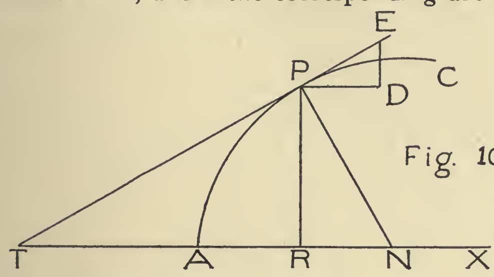


Fig. 10

$$DE : DP :: PR : TR;$$

that is,

$$dy : dx :: y : TR,$$

or

$$TR = \frac{ydx}{dy}. \qquad (1)$$

$$TP^2 = PR^2 + TR^2;$$

that is,

$$TP^2 = y^2 + \frac{y^2dx^2}{dy^2}$$

or

$$TP = \frac{y}{dy}(dx^2 + dy^2)^{\frac{1}{2}}. \qquad (2)$$

$$DP : DE :: PR : RN;$$

that is,

$$dx : dy :: y : RN$$

or

$$RN = \frac{ydy}{dx}. \qquad (3)$$

$$PN^2 = PR^2 + RN^2;$$

that is, $$PN^2 = y^2 + y^2 \frac{dy^2}{dx^2}$$

or $$PN = \frac{y}{dx} (dx^2 + dy^2)^{\frac{1}{2}}. \tag{4}$$

1. Hence *the length of the subtangent to any point of a curve is equal to the ordinate into the rate of the abscissa divided by the rate of the ordinate.*

2. *The length of the tangent to any point of a curve is equal to the ordinate divided by the rate of the abscissa, into the square root of the sum of the squares of the rates of the abscissa and ordinate.*

3. *The length of the subnormal to any point of a curve is equal to the ordinate into its rate divided by the rate of the abscissa.*

4. *The length of the normal to any point of a curve is equal to the ordinate divided by the rate of the abscissa, into the square root of the sum of the squares of the rates of the abscissa and ordinate.*

The tangent TP may also be obtained thus:

$$DE : PE :: PR : TP;$$

that is, $$dy : dz :: y : TP$$

or $$TP = y \frac{dz}{dy}. \tag{5}$$

Likewise, for the normal PN,

$$DP : PE :: PR : PN;$$

that is, $$dx : dz :: y : PN$$

or $$PN = y \frac{dz}{dx}. \tag{6}$$

In the application of these formulas to any particular curve, the value of $\frac{dx}{dy}$ or $\frac{dy}{dx}$, obtained from the equation of the curve by passing to the rate, must be substituted in each of them. The result will be true for all points of the curve; then, by substituting therein the values of x and y for any particular point of the curve, we can find the value of the subtangent, tangent, subnormal, and normal for that point.

41. To apply the formulas of the preceding articles to lines of the second order, whose general equation is

$$y^2 = ax^2 + bx + c,$$

and its rate

$$\frac{dy}{dx} = \frac{2ax + b}{2y} = \frac{2ax + b}{2\,(ax^2 + bx + c)^{1/2}}.$$

Substituting the value of $\frac{dy}{dx}$ in formulas (1), (2), (3), and (4) of Art. 40, will give

$$TR = y\frac{dx}{dy} = \frac{2(ax^2 + bx + c)}{2ax + b}$$

$$TP = \frac{y}{dy}\,(dx^2 + dy^2)^{1/2} =$$

$$\sqrt{\{ax^2 + bx + c + 4\,(\frac{ax^2 + bx + c}{2ax + b})^2\}}$$

$$RN = y\frac{dy}{dx} = \frac{2ax + b}{2}$$

$$PN = \frac{y}{dx}(dx^2 + dy^2)^{1/2} = \sqrt{\{ax^2 + bx + c + \frac{1}{4}(2ax + b)^2\}}.$$

By giving proper values to a, b, and c, these formulas will become applicable to any line of the second order.

In the case of the parabola, $a = 0$, $b = 2p$, and $c = 0$; therefore

$$TR = 2x$$
$$TP = (2px + 4x^2)^{1/2}$$
$$RN = p$$
$$PN = (2px + p^2)^{1/2}.$$

EXAMPLES

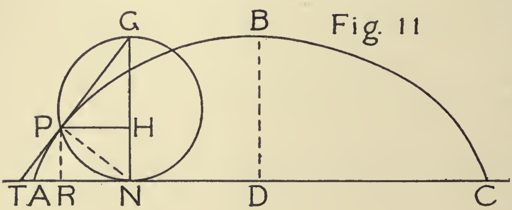
The major axis of an ellipse is 40 inches and the minor axis is 20 inches. What are the lengths of the tangent, subtangent, normal, and subnormal?

The transverse axis of a hyperbola is 6 inches and the conjugate axis is 4 inches. What is the length of the subnormal corresponding to an abscissa of 9 inches, the equation being $y^2 = \frac{B^2}{A^2} x^2 - B^2$?

The Cycloid

42. If a circle, NPG, be rolled along a straight line AC, any point P of its circumference will describe a curve called a cycloid. The circle NPG is called the generating circle and the point P is called the generating point.

The line AC is equal to the circumference of the generating circle and is called the base of the cycloid, and the line BD, drawn perpendicular to it at its middle point, is called the axis of the cycloid and is equal to the diameter of the generating circle.



In determining the equation of the cycloid, take the origin of the coördinates at A and suppose that the generating point has described the arc AP; then if N be the point at which the generating circle touches the base, AN will be equal to the arc NP.

Draw NG, the diameter of the generating circle, PR perpendicular to the base, and PH parallel to it; then PR will be equal to NH which is the versed sine of the arc NP.

Let $NG = 2r$, $AR = x$, and $PR = HN = y$; then

$$RN = PH = (NH \cdot HG)^{\frac{1}{2}} = (y \cdot \overline{2r - y})^{\frac{1}{2}} = (2ry - y^2)^{\frac{1}{2}};$$

also $\qquad x = AR = AN - RN = \text{arc } NP - PH.$

Therefore, since NP is the arc whose versed sine is NH or y (that is, $NP = \text{vers}^{-1} y$),

$$x = \text{vers}^{-1} y - (2ry - y^2)^{\frac{1}{2}} \qquad (1)$$

which is the transcendental equation of the cycloid.

The rate of $\text{vers}^{-1} y$ is $\frac{rdy}{(2ry - y^2)^{\frac{1}{2}}}$ and that of $(2ry - y^2)^{\frac{1}{2}}$ is $\frac{rdy - ydy}{(2ry - y^2)^{\frac{1}{2}}}$; therefore

$$dx = \frac{rdy}{(2ry - y^2)^{\frac{1}{2}}} - \frac{rdy - ydy}{(2ry - y^2)^{\frac{1}{2}}}$$

or
$$dx = \frac{ydy}{(2ry - y^2)^{\frac{1}{2}}}, \qquad (2)$$

which is the ratal equation of the cycloid.

43. Dividing both members of (2) of the preceding article by dy, gives

$$\frac{dx}{dy} = \frac{y}{(2ry - y^2)^{\frac{1}{2}}};$$

then, by substituting this value of $\frac{dx}{dy}$ in formulas (1), (2), (3), and (4) of Art. 40, the values of subtangent, tangent, subnormal, and normal for any point of the cycloid are as follows:

$$TR = \frac{y^2}{(2ry - y^2)^{\frac{1}{2}}}$$

$$TP = \frac{y\,(2ry)^{\frac{1}{2}}}{(2ry - y^2)^{\frac{1}{2}}}$$

$$RN = (2ry - y^2)^{\frac{1}{2}}$$

$$PN = \sqrt{(2ry)}.$$

The Logarithmic Curve

44. The logarithmic curve takes its name from the property that, when referred to rectangular axes, any abscissa is equal to the logarithm of the corresponding ordinate; hence the equation of the curve is

$$x = \log y.$$

If a represents the base of one system of logarithms, and b that of another, then

$$a^x = y \text{ and } b^x = y;$$

from which it is evident that for every different base the same value of x will give a different value for y; that is, every different logarithmic base will give a different logarithmic curve.

If we make $x = 0$, $y = 1$; therefore, since this relation is independent of the base of the system, it follows that every logarithmic curve will intersect the axis of ordinates at a distance from the origin equal to unity.

From
$$a^x = y$$
the curve can be described by points even without the aid of a table of logarithms; thus

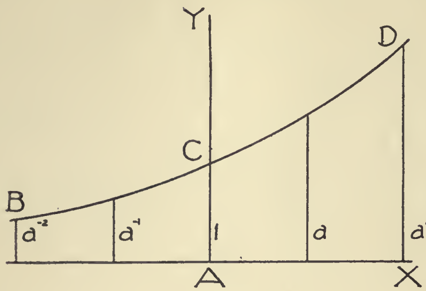


Fig. 12

$$x=0,\quad y=1;$$
$$x=1,\quad y=a;$$
$$x=2,\quad y=a^2;$$
$$x=-1,\quad y=a^{-1};$$
$$x=-2,\quad y=a^{-2},\text{ etc.}$$

Then if the origin is at A (see Fig. 12), BCD will be the curve.

Resuming the equation of the curve,

$$x=\log y;$$

then, if M represents the modulus of the system, the rate is

$$dx=\frac{Mdy}{y}$$

or

$$\frac{dx}{dy}=\frac{M}{y}.$$

Substituting this value of $\frac{dx}{dy}$ in formula (1) of Art. 40 gives

$$TR=y\frac{dx}{dy}=M.$$

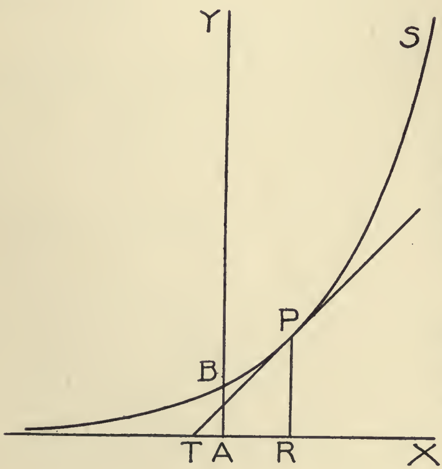


Fig. 13

Hence *the subtangent of the logarithmic curve is constant and equal to the modulus of the system in which the logarithms are taken* (see Fig. 13).

In the Naperian system $M=1$; consequently the subtangent of the curve in this system is equal to unity or AB.

If the value of $\frac{dx}{dy}$ be substituted in formulas (2), (3), and (4) of Art. 40, then

$$TP = (y^2 + M^2)^{\frac{1}{2}}$$

$$RN = \frac{y^2}{M}$$

$$PN = \frac{y}{M}(y^2 + M^2)^{\frac{1}{2}}.$$

Asymptotes

45. An asymptote is a right line which continually approaches a curve and becomes tangent to it only at an infinite distance from the origin of the coördinates.

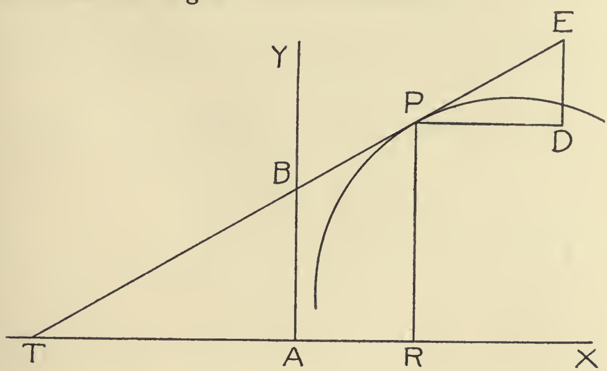


Fig. 14

Let A be the origin of the coördinates (see Fig. 14), and let TE be tangent to the curve at P; then, since the subtangent TR is equal to $y\frac{dx}{dy}$ and the abscissa $AR = x$,

$$AT = TR - AR$$

or

$$AT = y\frac{dx}{dy} - x. \qquad (1)$$

Also $\qquad DP : DE :: AT : AB$

or, since DP is represented by dx, DE by dy, and

$$AT = y\frac{dx}{dy} - x$$

$$dx : dy :: y\frac{dx}{dy} - x : AB,$$

whence $$AB = y - x\frac{dy}{dx}. \qquad (2)$$

If, when x and y become infinite, both of the expressions (1) and (2) also become infinite, it is evident the curve has no asymptote; but if either one or both of the expressions reduce to a finite quantity, it may be inferred that the curve has an asymptote.

If both expressions are finite, the asymptote will be inclined to both the coördinate axes; if one becomes finite and the other infinite, the asymptote will be parallel to one of the coördinate axes; if both become zero, the asymptote will pass through the origin of coördinates.

The general equation of lines of the second order is

$$y^2 = ax^2 + bx + c.$$

Passing to the rate,

$$2ydy = (2ax + b)\,dx,$$

whence $$y\frac{dx}{dy} = \frac{2y^2}{2ax + b}$$

or, substituting for $2y^2$ its value,

$$y\frac{dx}{dy} = \frac{2ax^2 + 2bx + 2c}{2ax + b};$$

it will be found that

$$x\frac{dy}{dx} = \frac{2ax^2 + bx}{2\,(ax^2 + bx + c)^{\frac{1}{2}}}.$$

Substituting these values of $y\frac{dx}{dy}$ and $x\frac{dy}{dx}$ in (1) and (2) gives

$$AT = \frac{2ax^2 + 2bx + 2c}{2ax + b} - x = \frac{bx + 2c}{2ax + b}, \qquad (3)$$

$$AB = (ax^2 + bx + c)^{\frac{1}{2}} -$$

$$\frac{2ax^2 + bx}{2\,(ax^2 + bx + c)^{\frac{1}{2}}} = \frac{bx + 2c}{2\,(ax^2 + bx + c)^{\frac{1}{2}}}. \qquad (4)$$

The equation of the parabola is

$$y^2 = 2px;$$

hence in $y^2 = ax^2 + bx + c$, $a = 0$, $b = 2p$, and $c = 0$; therefore (3) and (4) become

$$AT = \frac{2px}{2p} = x$$

and $$AB = \frac{2px}{2\sqrt{(2px)}} = \frac{1}{2}\sqrt{(2px)}.$$

Making x infinite, these equations become infinite also; therefore the parabola has no asymptote.

In the equation of the circle and ellipse, a in

$$y^2 = ax^2 + bx + \mathrm{c}$$

is negative; consequently AB becomes imaginary when x is infinite; therefore neither the circle nor the ellipse has an asymptote.

The equation of the hyperbola is

$$y^2 = \frac{B^2}{A^2}x^2 - B^2;$$

hence $b = 0$ in $y^2 = ax^2 + bx + c$; therefore (3) and (4) become

$$AT = \frac{2c}{2ax} = \frac{c}{ax}$$

and $$AB = \frac{2c}{2(ax^2 + c)^{1/2}} = \frac{c}{(ax^2 + c)^{1/2}}.$$

When x is infinite both of these equations become equal to zero; hence the hyperbola has asymptotes, one to either branch of the curve, both of which pass through the origin of coördinates.

The equation of the logarithmic curve is

$$x = \log y.$$

Passing to the rate

$$dx = \frac{Mdy}{y},$$

whence $$y\frac{dx}{dy} = M \text{ and } x\frac{dy}{dx} = \frac{xy}{M}.$$

Substituting these values in (1) and (2) gives

$$AT = y\frac{dx}{dy} - x = M - x$$

and $$AB = y - \frac{dy}{dx} = y - \frac{xy}{M}.$$

When $y = 0$, x is negative and infinite; but when x is negative and infinite (5) and (6) become

$$AT = \infty \text{ and } AB = 0;$$

hence the axis of abscissas of the logarithmic curve is an asymptote to that branch of the curve which lies on the left of the origin of the coördinates. (See Fig. 13 in Art. 44.)

Rates of Arc, Area, Surface, and Volume of Revolution

46. For the rate of an arc, see Art. 30, in which the formula thereof will be found, viz.,

$$dz = (dx^2 + dy^2)^{\frac{1}{2}},$$

z being the arc, x the abscissa, and y the ordinate.

Hence *the rate of an arc is equal to the square root of the sum of the square of the rates of the coördinates.*

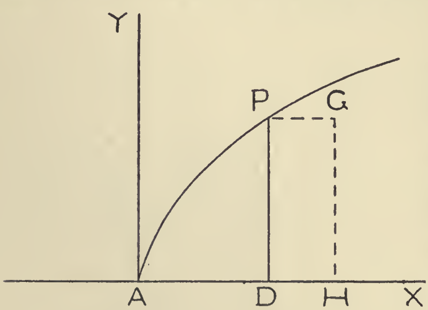


Fig. 15

47. Let x represent the abscissa AD (see Fig. 15), y the ordinate DP, and let PG or DH be represented by dx, the rate of x; then the rate of increase of the area APD will be represented by ydx. Therefore, if the area of APD be represented by A, then

$$dA = ydx.$$

Hence *the rate of the area of a segment of a curve is equal to the ordinate into the rate of the abscissa.*

48. Let x represent the abscissa AD, y the ordinate DP, z the arc AP, and S the surface of revolution made by the arc AP in revolving round AD or the axis of X; then the point P of the curve APC will describe a circle whose radius is y and consequently its circumference $2\pi y$. Now it is evident that if $2\pi y$ be multiplied by dz (which equals PT) the rate at which z, or the arc APC, is increasing at P, then

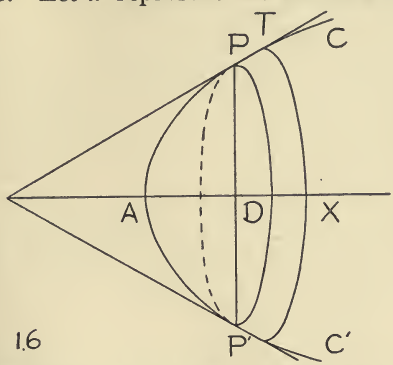


Fig. 16

$$dS = 2\pi y dz,$$

or, substituting for dz its value from Art. 46,

$$dS = 2\pi y (dx^2 + dy^2)^{\frac{1}{2}}.$$

Hence *the rate of the surface of the volume of revolution of an arc of a curve is equal to the circumference of a circle whose radius is the ordinate of the arc, multiplied by the rate of the arc.*

49. Let x represent any abscissa, as AD, y the correspond-

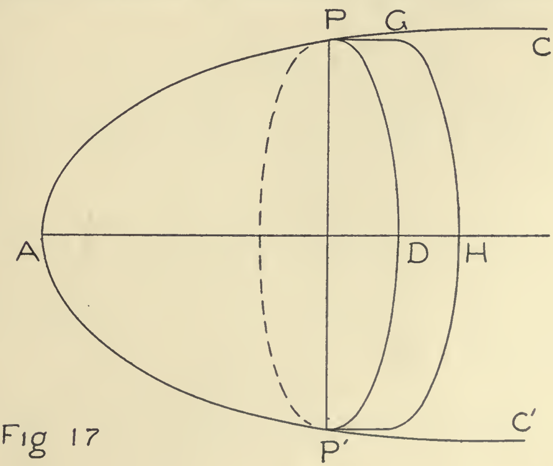


Fig 17

ing ordinate DP, and let dx be represented by $DH = PG$; then dx multiplied by the area of the circle described by the point P, in revolving round the axis of X, is equal to the rate at which the volume of revolution is increasing when the arc is AP, and consequently the ordinate is DP. Therefore, since $DP = y$, πy^2 is equal to the area of the circle described by the point P; consequently, if V represents the volume of revolution generated by the arc AP in revolving round the axis of X, then, since $PG = dx$,

$$dV = \pi y^2 dx.$$

Hence *the rate of the volume of revolution of an arc of a curve is equal to the area of the circle whose radius is the ordinate of the arc, into the rate of the abscissa.*

EXAMPLES

Determine the rate of an arc of the circle whose equation is

$$y^2 = r^2 - x^2.$$

Passing to the rate,

$$ydy = - xdx \text{ or } dy^2 = \frac{x^2dx^2}{y^2},$$

but, by Art. 46, $\quad dz = (dx^2 + dy^2)^{\frac{1}{2}};$
therefore

$$dz = (dx^2 + \frac{x^2dx^2}{y^2})^{\frac{1}{2}} \text{ or } dz = \frac{dx}{y}(x^2 + y^2)^{\frac{1}{2}}.$$

But, since $y = (r^2 - x^2)^{\frac{1}{2}}$ or $(x^2 + y^2)^{\frac{1}{2}} = r$, then

$$dz = \frac{rdx}{(r^2 - x^2)^{\frac{1}{2}}}.$$

Determine the rate of the area of the parabola; also the rate of the surface and volume of revolution of the hyperbola.

Radius of Curvature

50. Of curves tangent to each other and having a common tangent line at the point of contact, the one which departs most rapidly from the tangent line is said to have the greatest curvature. The curvature of a circle is measured by the angle formed by the radii drawn through the extremities of an arc of a given length.

Let R and R' represent the radii of two circles, a the length of a given arc measured on the circumference of each, c the angle formed by the radii drawn through the extremities of the arc of the one having radius R, and c' the angle similarly formed by the radii of the one having radius R'; then

$$2\pi R : 360^\circ :: a : c \text{ and } 2\pi R' : 360^\circ :: a : c',$$

whence
$$c = \frac{360^\circ a}{2\pi R} \text{ and } c' = \frac{360^\circ a}{2\pi R'},$$

therefore
$$c : c' :: \frac{360^\circ a}{2\pi R} : \frac{360^\circ a}{2\pi R'} \text{ or } c : c' :: \frac{1}{R} : \frac{1}{R'}.$$

Hence *the curvature in two different circles varies inversely as their radii.*

Make TNG a tangent line to the curve ANC, touching at the point N, and NM a normal line thereto (see figure); then

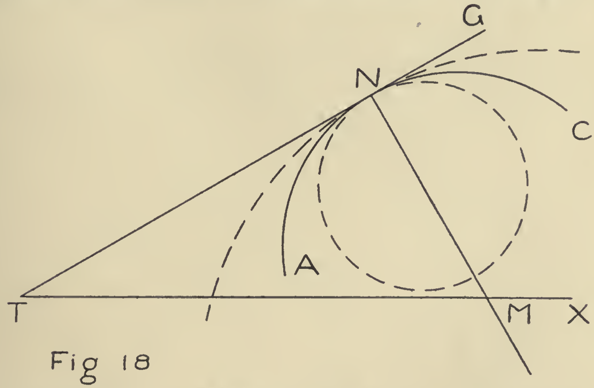


Fig 18

the circumference of every circle having its center in NM, which may be described through the point N, will touch at N both the curve ANC and the tangent line TNG.

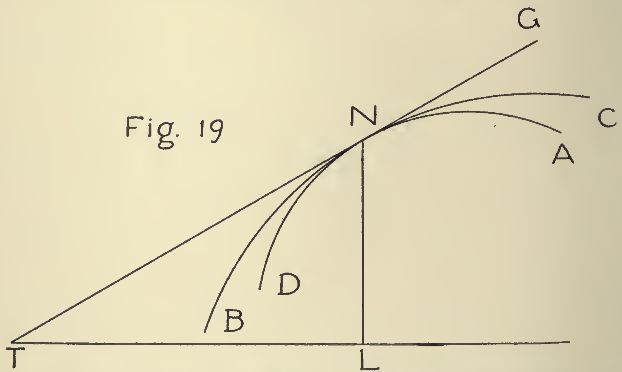
Now it is obvious that the circumference of any such circle which has a greater curvature than the curve ANC will depart more rapidly from the tangent line than ANC and consequently will fall wholly within ANC; but any circumference which has a less curvature than ANC will depart less rapidly from the tangent line than ANC and consequently will fall between it and TNG. Hence, since there may be circumferences of both less and greater curvature than ANC, it follows that, with a center in the line NM, a circumference may be described through the point N whose curvature will correspond with that of the curve ANC at N—that is, which will depart from the tangent line at N at the same rate as the curve ANC.

The circle, the curvature of whose circumference corresponds with that of any curve at any point, is called the *osculatory circle* or *circle of curvature,* and its radius the *radius of curvature* of the curve.

51. Let the curves BNC and DNE be tangent to each other at N, and draw TNG a tangent line to both, touching at N (see figure); also let $y=f(x)$ represent the curve BNC and $y'=f'(x')$ that of DNE. Draw the ordinate LN; then, since LN is common to both curves, $y=y'$ for the point N; also, since the angle LTN and consequently the tangent of the angle of tangency for the point N are common to both curves, $\frac{dy}{dx}=\frac{dy'}{dx'}$. These two conditions, $y=y'$ and $\frac{dy}{dx}=\frac{dy'}{dx'}$, existing, the curves are said to have a *contact of the first order.*

If at the point N the second ratal coefficients of the equations of the two curves are also equal (that is, if $\frac{d^2y}{dx^2}=\frac{d^2y'}{dx'^2}$)

Fig. 19



there will be a so-called *contact of the second order.* This is evident since either $\frac{d^2y}{dx^2}$ or $\frac{d^2y'}{dx'^2}$ is the same as the rate of variation of the tangent of the angle LTN (the angle of tangency); consequently both curves depart from the tangent line TNG at the same rate.

If, in addition, the third ratal coefficients of the equations of the curves are equal (that is, if $\frac{d^3y}{dx^3}=\frac{d^3y'}{dx'^3}$) the curves will have a *contact of the third order*, and so on for any order of of contact.

Now if BNC be given in species, magnitude, and position, and DNE in species only, then the constants which enter $y = f(x)$ will be fixed and determinate, while those which enter $y' = f'(x')$ will be entirely arbitrary, and therefore their values may be made to answer as many independent conditions as there are constants. Hence for a contact of the first order, $y = f(x)$ must contain at least two constants; for a contact of the second order, three constants; for a contact of the third order, four constants, and so on.

In the most general equation of the straight line, which is

$$y = ax + b,$$

there are two constants, a and b; therefore the straight line can have only a contact of the first order.

The most general equation of the circle is

$$(y - b)^2 = R^2 - (x - a)^2,$$

which contains three constants, a, b, and R; therefore the circle can have a contact of the second order.

In the general equation of the parabola,

$$\{(y - b) \cos v - (x - a) \sin v\}^2 =$$
$$2p \{(y - b) \sin v + (x - a) \cos v\},$$

there are four constants, a, b, v, and p; therefore the parabola can have a contact of the third order.

In the general equation of the ellipse or hyperbola there are five constants; therefore either the ellipse or hyperbola can have a contact of the fourth order.

The curve which has a higher order of contact with a given curve than can be found for any other curve of the same species is called the *osculatrix* of that species.

52. The general equation of the circle is

$$(y - b)^2 = R^2 - (x - a)^2. \qquad (1)$$

Passing to the rate twice, under the supposition that neither x nor y varies uniformly—that is, that neither dx nor dy is constant—then

$$(y - b)\, dy = -(x - a)\, dx$$

and $$(y - b)\, d^2y + dy^2 = -(x - a)\, d^2x - dx^2.$$

From these two equations the following are found:

$$y - b = -\frac{(dx^2 + dy^2)\, dx}{dx d^2y - dy d^2x}$$

and $$x-a=\frac{(dx^2+dy^2)\,dy}{dxd^2y-dyd^2x}.$$

Substituting these values of $y-b$ and $x-a$ in (1) will give

$$\frac{(dx^2+dy^2)^2\,dx^2}{(dxd^2y-dyd^2x)^2}=R^2-\frac{(dx^2+dy^2)^2\,dy^2}{(dxd^2y-dyd^2x)^2}$$

$$R^2=\frac{(dx^2+dy^2)^2\,dx^2}{(dxd^2y-dyd^2x)^2}+\frac{(dx^2+dy^2)^2\,dy^2}{(dxd^2y-dyd^2x)^2};$$

therefore $$R^2=\frac{(dx^2+dy^2)^3}{(dxd^2y-dyd^2x)^2}$$

or $$R=\pm\frac{(dx^2+dy^2)^{3/2}}{(dxd^2y-dyd^2x)}, \qquad (2)$$

which is the general expression for the value of the *radius of the osculatory circle.*

If dx be constant, $d^2x=0$ and (2) becomes

$$R=\pm\frac{(dx^2+dy^2)^{3/2}}{dxd^2y}, \qquad (3)$$

which is the expression for the value of the radius of the osculatory circle applied to curves referred to rectangular coördinates in which the abscissa is supposed to vary uniformly.

Hence, in order to find the radius of curvature for any particular curve, the first and second rates of its equation must be taken and the values of dx, dy, d^2y obtained and substituted in (3).

If z represents the arc, then $(dx^2+dy^2)^{1/2}=dz$; substituted in (3), this gives

$$R=\pm\frac{dz^3}{dxd^2y}. \qquad (4)$$

It has been shown in Art. 37 that y and $\frac{d^2y}{dx^2}$, consequently y and d^2y, have contrary signs when the curve is concave toward the axis of abscissas and like signs when convex; therefore, if we wish the radius of curvature and the ordinate of the curve to have like signs, we must employ the minus sign in (2), (3), and (4) when the curve is concave toward the axis of abscissas and the plus sign when convex.

If P and P' be any two points in the given curve $APP'B$, r the radius of the osculatory circle of the point P, and r' the radius of the point P' (see figure) then curvature at P : curvature at $P' :: \frac{1}{r} : \frac{1}{r'}$: that is, the curvature at different points of a curve varies inversely as the radii of the osculatory circles.

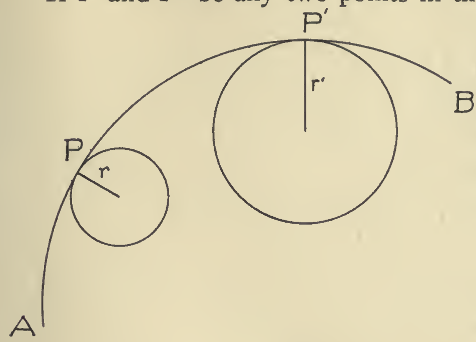


Fig. 20

53. The general equation of lines of the second order is

$$y^2 = ax^2 + bx + c \qquad (1)$$

and its rate

$$dy = \frac{(2ax + b)\,dx}{2y}; \qquad (2)$$

therefore

$$dx^2 + dy^2 = dx^2 + \frac{(2ax+b)^2\,dx^2}{4y^2} = \frac{[4y^2 + (2ax+b)^2]\,dx^2}{4y^2}$$

or

$$(dx^2 + dy^2)^{3/2} = \frac{[4y^2 + (2ax+b)^2]^{3/2}\,dx^3}{8y^3}. \qquad (3)$$

The rate of (2) is

$$d^2y = \frac{2aydx^2 - (2ax+b)\,dxdy}{2y^2},$$

whence, since $dxdy = \frac{(2ax+b)\,dx^2}{2y}$ [see (2)], by substituting it and reducing,

$$d^2y = \frac{[4ay^2 - (2ax+b)^2]\,dx^2}{4y^3}, \qquad (4)$$

but [see (1)]

$$4ay^2 = 4a^2x^2 + 4abx + 4ac = (2ax+b)^2 + 4ac - b^2$$

or

$$4ay^2 - (2ax+b)^2 = 4ac - b^2;$$

therefore, substituting this in (4) and multiplying by dx,

$$dxd^2y = \frac{(4ac - b^2)\, dx^3}{4y^3}. \qquad (5)$$

Substituting (3) and (5) in (3) of Art. 52, then

$$R = \pm \frac{[4y^2 + (2ax + b)^2]^{3/2}}{2\,(4ac - b^2)},$$

or, substituting for y its value,

$$R = \pm \frac{[4\,(ax^2 + bx + c) + (2ax + b)^2]^{3/2}}{2\,(4ac - b^2)}, \qquad (6)$$

which is the general expression for the radius of curvature of lines of the second order for any abscissa x.

If both numerator and denominator of (6) be divided by 8, then

$$R = \pm \frac{\{ax^2 + bx + c + \frac{1}{4}\,(2ax + b)^2\}^{3/2}}{ac - \frac{1}{4}\,b^2}. \qquad (7)$$

The numerator of this value of R is the cube of the normal [see (6), Art. 40]; therefore, since the denominator is constant, it is evident from Art. 52 that *the radii of curvature at different points of lines of the second order are to each other as the cubes of the corresponding normals.*

If the origin of coördinates is at the vertex of the transverse axis, $c = 0$; consequently, using the minus sign, (6) becomes

$$R = \frac{\{4\,(ax^2 + bx) + (2ax + b)^2\}^{3/2}}{2b^2},$$

which, when $x = 0$, reduces to

$$R = \frac{1}{2}\,b.$$

In this case b is the parameter of the curve; therefore the radius of curvature at the vertex of the transverse axis of lines of the second order is equal to half the parameter of the curve.

In the case of the parabola whose equation is

$$y^2 = 2px,$$

$a=0$, $c=0$, $b=2p$; therefore, substituting these values in (7) and using the minus sign,

$$R=\frac{(2px+p^2)^{3/2}}{p^2},$$

which is the general value of the radius of curvature for any point of the parabola. If $x=0$, then $R=p$, the radius of curvature at the vertex of the axis.

In the case of the ellipse whose equation is

$$y^2=B^2-\frac{B^2}{A^2}x^2,$$

$a=-\frac{B^2}{A^2}$, $b=0$, and $c=B^2$; therefore, substituting these values in (6), reducing and using the minus sign,

$$R=\frac{(A^4-A^2x^2+B^2x^2)^{3/2}}{A^4B},$$

which is the general value of the radius of curvature for any point of the ellipse. If $x=0$, then $R=\frac{A^2}{B}$, which is the radius of curvature at the vertex of the minor axis. If $x=A$, then $R=\frac{B^2}{A}$, which is the radius of curvature at the vertex of the major axis.

Taking the equation of the logarithmic curve,

$$x=\log y,$$

and passing to the rate twice,

$$dx=M\frac{dy}{y} \text{ or } dy=\frac{ydx}{M}$$

and

$$d^2y=\frac{dxdy}{M},$$

whence

$$(dx^2+dy^2)^{3/2}=\frac{dy^3}{y^3}(M^2+y^2)^{3/2}.$$

Substituting the values of dxd^2y and $(dx^2+dy^2)^{3/2}$ in

(3) of Art. 52 and using the plus sign, for any point of the logarithmic curve

$$R = \frac{(M^2 + y^2)^{3/2}}{My}.$$

When y is equal to the modulus of the system of logarithms employed,

$$R = 2M\sqrt{2}.$$

From the ratal equation of the cycloid (Art. 42)

$$dx = \frac{ydy}{(2ry - y^2)^{1/2}}. \qquad (1)$$

Passing to the rate and reducing,

$$0 = (yd^2y + dy^2)(2ry - y^2)^{1/2} - \frac{ydy^2(r - y)}{(2ry - y^2)^{1/2}} =$$

$$(2ry - y^2)\,yd^2y + rydy^2;$$

whence

$$d^2y = -\frac{rdy^2}{2ry - y^2} \text{ or } dxd^2y = -\frac{rydy^3}{(2ry - y^2)^{3/2}}.$$

It will also be found that

$$dx^2 + dy^2 = \frac{y^2dy^2}{2ry - y^2} + dy^2 =$$

$$\frac{y^2dy^2 + 2rydy^2 - y^2dy^2}{2ry - y^2} = \frac{2rydy^2}{2ry - y^2};$$

therefore

$$(dx^2 + dy^2)^{3/2} = \frac{2rydy^3\sqrt{(2ry)}}{(2ry - y^2)^{3/2}}.$$

Substituting the values of dxd^2y and $(dx^2 + dy^2)^{3/2}$ in (3) of Art. 52 and using the minus sign will give

$$R = 2\sqrt{(2ry)};$$

but the normal is equal to $\sqrt{(2ry)}$ by Art. 43; hence the radius of curvature for any point of the cycloid is equal to twice the normal at the point of contact.

Evolutes and Involutes

54. An *evolute* is a curve from which a thread is supposed to be unwound or evolved, its extremity describing another curve called an *involute.*

Thus, let a thread be wrapped about the curve $BCC'D$ (Fig. 21); then, if the thread be kept tight and unwound from $BCC'D$, its extremity, commencing at A, will describe the curve $APP'S$. The curve $BCC'D$ is called the evolute of the curve $APP'S$ and $APP'S$ the involute of $BCC'D$.

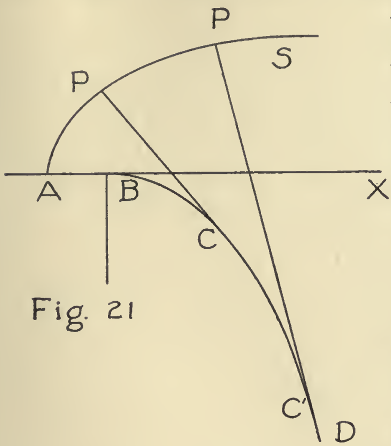


Fig. 21

From the manner in which the involute is generated it is evident that any portion of the thread, as CP, which is disengaged from the evolute is a tangent to it at C and perpendicular to the involute at P; also, that any point in the evolute, as C, may be considered as a center, and the line CP as the radius of a circle of whose circumference that portion of the involute curve at P is an arc.

The points B, C, C' are therefore centers, and the lines BA, CP, $C'P'$ the radii of circles of curvature of the points A, P, P' of the involute; hence any radius of curvature, as CP, is equal to AB plus the arc BC of the evolute.

The value of AB will depend upon the position of the point B, from which the arc of the evolute is estimated; but since AB is the radius of curvature of the involute at A, if A is the origin of the involute and B the corresponding origin of the evolute, B will be the center of the osculatory circle to the involute at its origin. Therefore, if the radius of curvature at the origin of the involute is equal to zero, A and B will coincide, and consequently AB will be equal to zero. If the involute is a curve of the second order, the radius of curvature at the vertex of the transverse axis is equal to half its parameter, $\frac{1}{2}b$, by Art. 53; consequently AB will be equal to $\frac{1}{2}b$, and B, the origin of the evolute, will be in the axis of abscissas AX.

Hence, since the center of any circle of curvature of the curve $APP'S$ is in the curve $BCC'D$, it follows that the equation representing the coördinates of the center of any circle of curvature of the involute will be the equation of the evolute.

Now the general equation of the circle, consequently of any circle of curvature, is

$$(y-b)^2 = R^2 - (x-a)^2, \qquad (1)$$

in which a and b are the coördinates of its center and x and y the coördinates of any points of its circumference; therefore a and b will represent the coördinates of any point, as C, of the evolute $BCC'D$, and its equation will be $b = f(a)$; also x and y will represent the coördinates of any corresponding point, as P, of the involute $APP'S$, and its equation will be $y = f(x)$.

Taking the rate of (1) twice

$$(y-b)\,dy = -(x-a)\,dx$$

and

$$dy^2 + (y-b)\,d^2y = -dx^2;$$

whence

$$b = y + \frac{dx^2 + dy^2}{d^2y} \qquad (2)$$

and

$$a = x - \frac{dy}{dx}\left(\frac{dx^2 + dy^2}{d^2y}\right). \qquad (3)$$

These are expressions for the values of the coördinates of the evolute in terms of the rates of the coördinates of the involute.

Hence, if we take the rate of the equation of the involute twice, $y = f(x)$, obtain the values of dx^2, dy^2, $\frac{dy}{dx}$, and d^2y, and then substitute them in (2) and (3), we shall have two new equations, expressing the values of a and b, the coördinates of the evolute, in terms of x and y, the coördinates of the involute.

Finally, by combining the equations thus found with the equation of the involute and eliminating x and y, an equation will be obtained containing only a and b, which will be the equation of the evolute.

Taking the equation of the common parabola

$$y^2 = 2px,$$

and passing to the rate twice

$$ydy = pdx$$

and

$$dy^2 + yd^2y = 0;$$

whence

$$dx^2 + dy^2 = \left(\frac{p^2}{y^3} + 1\right) dx^2, \quad \frac{dy}{dx} = \frac{p}{y}, \text{ and } d^2y = -\frac{p^2}{y^3}\,dx^2.$$

Substituting these values in (2) and (3) and reducing give

$$b = -\frac{y^3}{p^2} \text{ or } b^2 = \frac{y^6}{p^4}, \text{ and } a = x + \frac{y^2}{p} + p.$$

Substituting $2px$ for y^2 in the last two equations, it is found that

$$b^2 = \frac{8x^3}{p}, \qquad (4)$$

and

$$a = 3x + p \text{ or } x = \frac{1}{3}(a - p).$$

Finally, substituting $\frac{1}{3}(a-p)$ for x in (4), then

$$b^2 = \frac{8}{27p}(a-p)^3, \qquad (5)$$

which is the equation of the evolute and shows it to be the semi-cubical parabola.

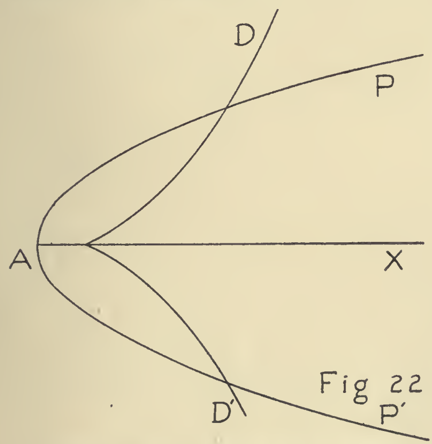


Fig 22

If we make $b = 0$, then $a = p$; hence, the evolute meets the axis of abscissas at a distance AB from the origin (Fig. 22) equal to half the parameter of the involute.

If the origin of the coördinates of the evolute be transferred from A to B, (5) becomes

$$b^2 = \frac{8a^3}{27p}.$$

Since every value of a gives two equal values of b with contrary signs, the curve is symmetrical with respect to the axis of abscissas; the evolute BD' corresponding to the part AP of the involute and BD to the part AP'.

From the equations relative to the cycloid, Art. 53, it is found that

$$dx^2 + dy^2 = \frac{2rydy^2}{2ry - y^2}, \quad \frac{dy}{dx} = \frac{(2ry - y^2)^{1/2}}{y},$$

and $$d^2y = -\frac{rdy^2}{2ry - y^2}.$$

Substituting these values in (2) and (3) of Art. 54 will give

$$b = -y \text{ and } a = x + 2\,(2ry - y^2)^{1/2},$$

whence $$y = -b \text{ and } x = a - 2\,(-2rb - b^2)^{1/2}.$$

Substituting these values of x and y in the transcendental equation of the cycloid (Art. 42) gives

$$a - 2\,(-2rb - b^2)^{1/2} = \text{vers}^{-1}\,(-b) - (-2rb - b^2)^{1/2},$$

or $$a = \text{vers}^{-1}\,(-b) + (-2rb - b^2)^{1/2},$$

which is the transcendental equation of the evolute of the cycloid, referred to the primitive axes and origin.

From the equation of the radius of curvature for the cycloid, $R = 2\sqrt{(2ry)}$ (see Art. 53), we have $R = 0$ when $y = 0$, and when $y = 2r = BD$, $R = 4r = A'B$; therefore the origin of the evolute is at A, and $A'D = BD$.

By transferring the origin of the coördinates of the evolute from A to A' and estimating the abscissas from the right toward the left, a new equation of the evolute is formed which will be found to be of the same form and to involve the same constants as the equation of the cycloid; hence the evolute of a cycloid is an equal cycloid—that is, the arc AA' is a facsimile of the arc AB, and $A'C$ of the arc CB.

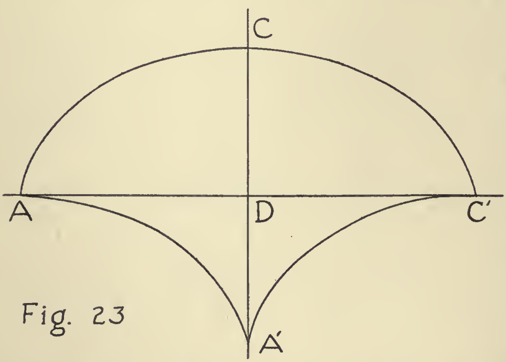


Fig. 23

Since the origin of the evolute is at A and the radius of curvature for the vertex B of the cycloid is $4r$, the length of the evolute AA' is $4r$; hence the length of the cycloid ABC is equal to $8r$, or four times the diameter of the generating circle.

EXAMPLES

Determine the length of the radius of curvature for a point in a parabola whose abscissa is four inches and ordinate six inches.

Determine the length of the radius of curvature for a point in an ellipse, whose abscissa is 16 inches, measured from the center, the semi-axes being 26 and 13 inches.

Determine the equation of the evolute of the equilateral hyperbola, its equation being $y^2 = x^2 - A^2$.

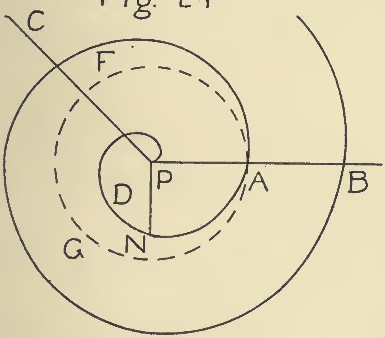
Determine the evolute of the spiral whose ratal equation is $dv = \frac{dr}{ar}(r^2 - a^2)^{\frac{1}{2}}$.

Curves Referred to Polar Coördinates

55. If the right line PC (Fig. 24) revolves uniformly around the point P, and if at the same time a point moves from P along the line PC at such a rate that at the first revolution of PC it will arrive at A, at the second at B, etc., the curve described by the point will be a spiral.

The point P about which the right line revolves is called the *pole*; the point which moves along the line PC and describes the curve is called the *generating point;* a straight line drawn from the point P or eye of the spiral, so called, to any point of the curve, as N, is called the *radius vector*, and each portion of the spiral described by the generating point, as PDA, AEB, is called a *spire*.

Fig. 24



With the pole as a center and PA (the distance which the generating point moves from P along PC during the first revolution of PC) as a radius, if the circle AFG be described, the angular motion of PC about the pole, consequently the radius vector, as PN, is measured by an arc of this circle, estimated from A.

Now, if r represents the radius vector and v the measuring arc estimated from A, it is evident that r is a function of v and may generally be represented by the equation,

$$r = av^n, \tag{1}$$

in which a and n are constants. The value of n depends upon the law which governs the motion of the generating point along the radius vector and the value of a upon the relation existing between a given value of r and the corresponding value of v.

If n is positive, the spiral represented by (1) commences at the pole, for when $v=0$, $r=0$. If n is negative, the equation becomes

$$r=av^{-n}; \tag{2}$$

consequently the spiral commences at an infinite distance from the pole, for when $v=0$, r is infinite, or when $r=0$, v is infinite.

When n is equal to unity, (1) becomes

$$r=av. \tag{3}$$

Now if $a=AP$, the circumference of the circle AFG will be $2a\pi$, which is the measuring arc for the first revolution of PC; therefore, since PA or a is then the radius vector,

$$a=a\cdot 2a\pi$$

whence

$$a=\frac{1}{2\pi}.$$

Substituting this value of a in (3) gives

$$r=\frac{v}{2\pi},$$

the equation of the spiral of Archimedes.

When n is equal to one half, (1) becomes

$$r=av^{\frac{1}{2}} \text{ or } r^2=a^2v,$$

which is the equation of the parabolic spiral, being of the same form as that of the parabola; for substituting y for r, $\sqrt{(2p)}$ for a, and x for v gives

$$y^2=2px. \tag{4}$$

With $2p$ as radius draw the circle ABC, divide its circumference into any number of equal parts, as six, and draw through its center P, the divisional lines DD', EE', FF'. With $\frac{1}{2}(2p+1)$, $\frac{1}{2}(2p+2)$, $\frac{1}{2}(2p+3)$, and $\frac{1}{2}(2p+4)$, (1, 2, 3, and 4 being values given x as in the construction of the parabola) as radii, draw the arcs Aa, AbC, Ac, and Ad, having their centers in the line AG; then with Pa, Pb, Pc, and Pd, as radii, draw the arcs aa', bb', cc', and dd', and the curve drawn

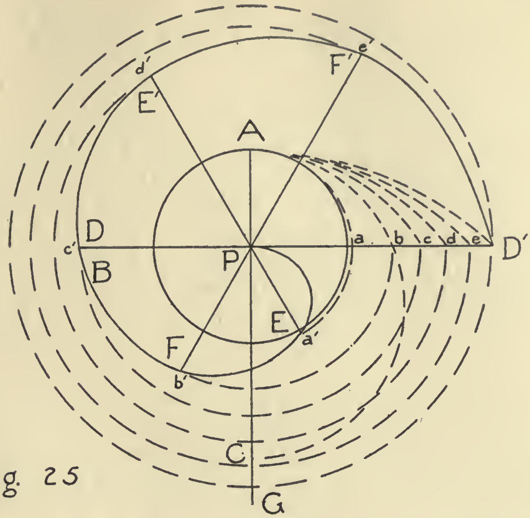


Fig. 25

from P through a', b', c', d' will be the required spiral. In proof, it will be seen that $AP:PC$ is equal to Pb^2 or Pb'^2 (see Euclid, proposition 35, Book III); hence, since $AP = 2p$ and $PC = x$, if y be represented by $Pb = Pb'$, then

$$y^2 = 2px.$$

Also, when $x = 0$, $y = 0$; therefore the spiral commences at P, its pole.

When n is equal to -1, (1) becomes

$$r = av^{-1} \text{ or } r = \frac{a}{v}. \tag{5}$$

The curve represented by this equation is called the hyperbolic spiral on account of its analogy to that of the hyperbola when referred to its center and asymptote.

With a as a radius draw the circle ABC and divide its circumference, $2a\pi$, into any number of equal parts, as six; then giving to v the values $2a\pi$, $\frac{5a\pi}{3}$, $\frac{4a\pi}{3}$, $a\pi$, etc., the corresponding values of r will be $\frac{1}{2\pi}$,

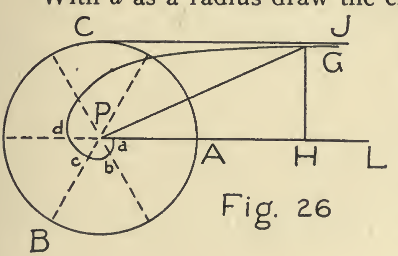


Fig. 26

$\frac{3}{5\pi}$, $\frac{3}{4\pi}$, $\frac{1}{\pi}$, etc. Let *Pa, Pb, Pc, Pd,* etc. represent these values of r; then the curve drawn through *a, b, c, d,* etc., will be the hyperbolic spiral.

Take any point in the spiral, as G, and draw GH perpendicular to PL; then $PG = r$ and the angle $GPH = v$; hence

$$GH = r \sin v,$$

or substituting for r its value from (5)

$$GH = \frac{a \sin v}{v}. \qquad (6)$$

Now it is evident that the smaller the value of v, the nearer will v and $\sin v$ approach equality and consequently the nearer will GH become equal to a; therefore, if CJ be drawn parallel to PL, CJ must be an asymptote to the spiral.

The equation of the logarithmic spiral, so called, is

$$a \log r = v. \qquad (7)$$

This spiral may be constructed as follows. With unity for radius draw the circle ABC; then, giving to r the values 1, 2, 3, etc., the corresponding values of v will be 0, $a \log 2$, $a \log 3$.

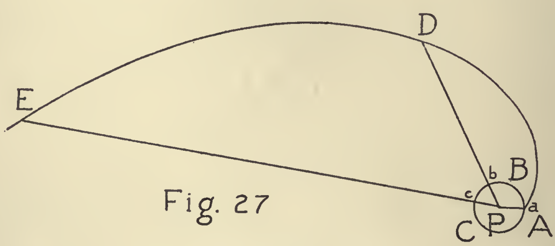


Fig. 27

Set off from A on the circumference of the circle these values of v, A, Ab, Ac; then to A and through b, c, draw PA, PD, PE, the values of r, and the curve drawn through A, D, E, will be the logarithmic spiral.

Since the relation between r and v is entirely arbitrary,

$$r = \frac{p}{1 + \cos v}, \qquad (8)$$

is the polar equation of the parabola, the pole being at the focus.

The polar equation of the ellipse, the pole being at one of the foci, is

$$r = \frac{p}{1 + e \cos v}. \qquad (9)$$

The polar equation of the hyperbola, the pole being at one of the foci, is

$$r = \frac{p}{1 + e \cos v}. \qquad (10)$$

In (8), (9), and (10) p represents half the parameter, e the eccentricity, and v the angle which the radius vector makes with the axis of X.

Tangents and Normals

56. The *subtangent* to a spiral is a line drawn from the pole perpendicular to the radius vector and limited by a tangent drawn through the extremity of the radius vector; the *tangent* is a line extending from the point of tangency to the subtangent; the *subnormal* is a line drawn from the pole to the foot of the normal; the *normal* is a line drawn perpendicular to the tangent and extending from the point of tangency to the subtangent extended.

Let r be any radius vector, as PN (see Fig. 28), v the measuring arc estimated from A, and z the corresponding arc

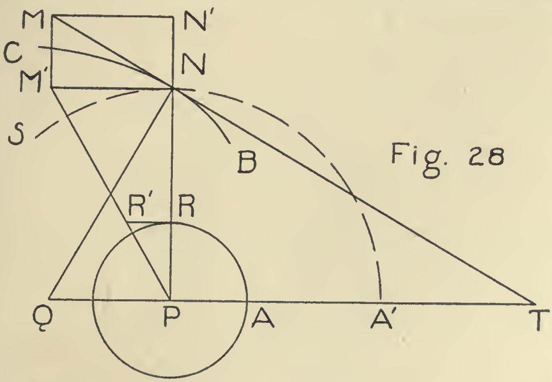


Fig. 28

of a spiral of which BNC is a section. Then, if dz be represented by NM, a tangent to BNC at N, and PN be extended to N, so that the angle $NN'M$ will be a right angle, it is evident that NN' will represent dr, the rate at which the radius vector PN is increasing. Now, draw NM' parallel to $N'M$ and MM'

parallel to NN'; also, with P as a center and PN as a radius, describe the circular arc $A'NS$. Then it will be seen that NM', tangent to $A'NS$ at N, will represent the rate at which the arc $A'N$ is increasing at N. Hence, since NM' represents the rate at which the arc $A'N$ is increasing, it is obvious that RR', tangent to the measuring circle at R, represents dv, the rate at which v, the measuring arc estimated from A, is increasing to correspond with NM or dz.

Therefore, since the triangles PRR' and PNM' are similar,

$$PR : RR' :: PN : NM',$$

or, making the radius of the measuring circle unity,

$$1 : dv :: r : NM',$$

whence

$$NM' = rdv. \qquad (1)$$

Again, from the similar triangles $MM'N$ and NPT, $MM' : NM' :: NP : PT$, or, since $NP = r$, $MM' = NN' = dr$, and from (1), $NM' = rdv$,

$$dr : rdv :: r : PT,$$

whence

$$PT = \frac{r^2dv}{dr}; \qquad (2)$$

but PT is the subtangent of the spiral, hence:

The length of the subtangent to any point of a spiral is equal to the square of the radius vector into the rate of the measuring arc, divided by the rate of the radius vector.

For the tangent TN,

$$TN^2 = PN^2 + PT^2—$$

that is,

$$TN^2 = r^2 + \frac{r^4dv^2}{dr^2}$$

or

$$TN = \frac{r}{dr}(dr^2 + r^2dv^2)^{\frac{1}{2}}. \qquad (3)$$

Hence *the length of the tangent to any point of a spiral is equal to the square root of the sum of the squares of the radius vector and subtangent.*

For the subnormal PQ,

$$PT : PN :: PN : PQ—$$

that is,

$$\frac{r^2dv}{dr} : r :: r : PQ$$

or

$$PQ = \frac{dr}{dv}. \qquad (4)$$

Hence *the length of the subnormal to any point of a spiral is equal to the rate of the radius vector divided by the rate of the measuring arc.*

For the normal QN,

$$QN^2 = PN^2 + PQ^2 -$$

that is,
$$QN^2 = r^2 + \frac{dr^2}{dv^2}$$

or
$$QN = (r^2 + \frac{dr^2}{dv^2})^{\frac{1}{2}}. \qquad (5)$$

Hence *the iength of the normal to any point of a spiral is equal to the square root of the sum of the squares of the radius vector and subnormal.*

The tangent of the angle of tangency of a spiral, PTN, since $PN = r$ and $PT = \frac{r^2 dv}{dr}$ from (2), is

$$\frac{PN}{PT} = \frac{dr}{rdv}. \qquad (6)$$

Hence *the tangent of the angle of tangency of a spiral is equal to the rate of the radius vector divided by the radius vector into the rate of the measuring arc.*

The tangent of the angle PNT is equal to $\frac{PT}{PN}$; but, since $PN = r$ and $PT = \frac{r^2 dv}{dr}$,

$$\frac{PT}{PN} = \frac{rdv}{dr}, \qquad (7)$$

which is the tangent of the angle the tangent line makes with the radius vector.

Of the general equation of spirals,

$$r = av^n,$$

the rate is
$$\frac{dr}{dv} = anv^{n-1} \text{ or } \frac{dv}{dr} = \frac{1}{anv^{n-1}}.$$

Substituting the value of $\frac{dr}{dv}$ or $\frac{dv}{dr}$, also av^n for r, in formulas (2), (3), (4), (5), (6), and (7), the result will be

$$PT = \frac{av^{n+1}}{n}$$

$$TN = (a^2n^{2n} + \frac{a^2v^{2n+2}}{n^2})^{\frac{1}{2}} = \frac{av^n}{n}(n^2 + v^2)^{\frac{1}{2}}$$

$$PQ = anv^{n-1}$$

$$QN = (a^2v^{2n} + a^2n^2v^{2n-2})^{\frac{1}{2}} = av^{n-1}(n^2 + v^2)^{\frac{1}{2}}$$

$$\frac{PN}{PT} = \frac{anv^{n-1}}{av} = \frac{n}{v}$$

$$\frac{PT}{PN} = \frac{v}{n}.$$

In the equation of the spiral of Archimedes, $r = \frac{v}{2\pi}$, $n = 1$, and $a = \frac{1}{2\pi}$. By substituting these values in the preceding formulas, the following are obtained:

$$PT = \frac{v^2}{2\pi},\ TN = \frac{v}{2\pi}(1 + v^2)^{\frac{1}{2}},\ PQ = \frac{1}{2\pi},$$

$$QN = \frac{1}{2\pi}(1 + v^2)^{\frac{1}{2}},\ \frac{PN}{PT} = \frac{1}{v},\ \frac{PT}{PN} = v.$$

If $v = 2\pi$—that is, if the tangent is drawn at the extremity of the arc generated in the first revolution of the radius vector—then

$$PT = 2\pi—$$

that is, PT is equal to the circumference of the measuring circle.

At the completion of m revolutions $v = 2m^2\pi$, and consequently $$PT = 2m^2\pi = m \cdot 2m\pi—$$
that is, at the completion of m revolutions the subtangent is equal to m times the circumference of the circle described with the radius vector of the mth revolution.

In the equation of the hyperbolic spiral, $r = av$, $n = -1$; therefore $$PT = -a.$$

Hence *the subtangent of the hyperbolic spiral is constant.*

From the equation of the logarithmic spiral,

$$v = \log r,$$

it will be found that $$\frac{rdv}{dr} = M;$$

but $\frac{rdv}{dr}$ [see (7)] represents the tangent of the angle made with the radius vector by a tangent line to the curve. Hence *the tangent of the angle which the tangent line makes with the radius vector is constant and equal to the modulus of the system of logarithms employed. In the Naperian system the modulus is unity; therefore, if v is the Naperian logarithm of r, the angle which the tangent line makes with the radius vector is* 45°.

Rate of the Arc and Area of Spirals

57. Let BNC in Fig. 29 be a section of a spiral, P the pole and TN a tangent to the curve at N. Draw PN, and NM' at right angles to PN; also extend TN to M and draw MM' so that $NM'M$ will be a right angle; then

$$NM^2 = M'M^2 + M'N^2.$$

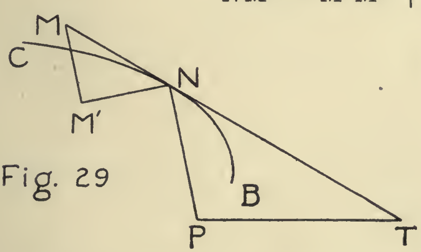


Fig. 29

But since NM represents dz; $M'M$, dr; and $M'N$, rdv [see (1) of Art. 51],

$$dz^2 = dr^2 + r^2dv^2 \text{ or } dz = (dr^2 + r^2dv^2)^{\frac{1}{2}}. \qquad (1)$$

Hence *the rate of an arc of a spiral is equal to the square root of the sum of the squares of the rate of the radius vector and of the product of the radius vector and the rate of the measuring arc.*

Let PN be a radius vector of the spiral $PBNC$ in Fig 30. Draw NM' and $M'P$, making the angle PNM' a right angle. Then, representing the area by A,

$$dA = \frac{1}{2} PN \cdot NM',$$

or, since $PN = r$ and, by (1) of Art. 51, $NM' = rdv$,

$$dA = \frac{1}{2} r^2dv. \qquad (2)$$

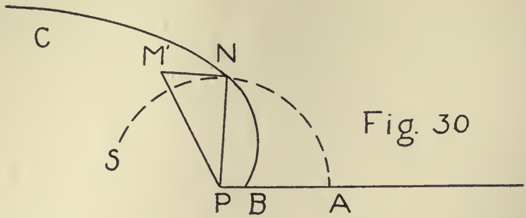


Fig. 30

This is evident from what has been shown in Art. 51; for, since NM' represents the rate at which the arc AN is increasing at N, it must also represent the rate at which the extremity of the radius vector is revolving when it arrives at N. Consequently the area of the triangle PNM' represents the rate at which the area of the spiral is increasing when the radius vector is PN.

Hence *the rate of the area of a spiral is equal to one-half the square of the radius vector into the rate of the measuring arc.*

EXAMPLES

Determine the rate of an arc of the parabolic spiral.

Determine the rate of the area of the hyperbolic spiral.

If the rate of the measuring circle of the Naperian logarithmic spiral is three, at what rate is the area of the spiral increasing when the radius vector is four?

Radius of Curvature for Spirals

58. Of the spiral PNS in Fig. 31, the subtangent $PT = \frac{r^2dv}{dr}$ (see (2) of Art. 56), the tangent $NT = \frac{r}{dr}(dr^2 + r^2dv^2)^{\frac{1}{2}}$ [see (3) of Art. 56], the radius vector $NP = r$, and $CN = R$, the radius of the osculatory circle AMN, NQ being normal to the spiral.

Join CP and draw DP parallel and BP perpendicular to NQ; then, since $BN = DP$,

$$CP^2 = CN^2 + NP^2 - 2CN \cdot DP$$

or

$$CP^2 = R^2 + r^2 - 2R \cdot DP;$$

but $DP = r \sin PND$, or, since $\sin PND$ is also equal to

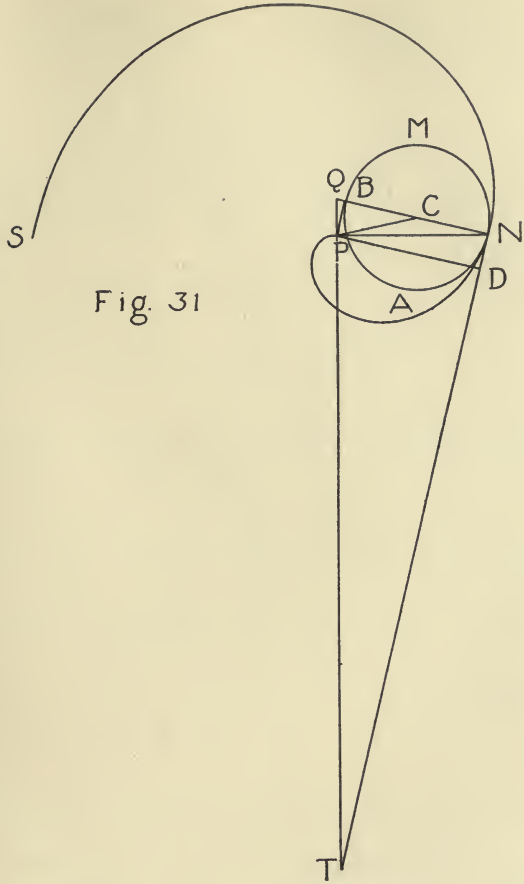


Fig. 31

$$\frac{PT}{NT}=\frac{rdv}{(dr^2+r^2dv^2)^{1/2}},\quad DP=\frac{r^2dv}{(dr^2+r^2dv^2)^{1/2}};\text{ therefore}$$

$$CP^2=R^2+r^2-\frac{2Rr^2dv}{(dr^2+r^2dv^2)^{1/2}}. \qquad (1)$$

Now the equation of the spiral is $r=av^n$, whence $v=\frac{r^{1/n}}{a^{1/n}}$; hence $dv=\frac{r^{(1-n)/n}dr}{na^{1/n}}$.

Substituting this value of dv in (1), then

$$CP^2=R^2+r^2-\frac{2Rr^{(1+n)/n}}{(r^{2/n}+n^2a^{2/n})^{1/2}}. \qquad (2)$$

Passing to the rate, CP and R being constant for any point of the circle AMP, and reducing,

$$R = \frac{(r^{2/n} + n^2 a^{2/n})^{3/2}}{r^{(3-n)/n} + n(1+n)\, a^{2/n} r^{(1-n)/n}}, \qquad (3)$$

which is the general value of the radius of curvature for all spirals represented by the equation $r = av^n$, in terms of the radius vector.

In the case of the spiral of Archimedes, $n = 1$, and (3) becomes

$$R = \frac{(r^2 + a^2)^{3/2}}{r^2 + 2a^2}.$$

For the logarithmic spiral, whose equation is $\log r = v$,

$$dv = \frac{Mdr}{r};$$

therefore, substituting this value of dv in (2), it will be found that

$$CP^2 = R^2 + r^2 - \frac{2RMr}{(M^2+1)^{1/2}}.$$

Passing to the rate and reducing,

$$R = \frac{r(M^2+1)^{1/2}}{M}.$$

If the Naperian system be used, $M = 1$, and $R = r\sqrt{2}$.

Determine the radius of curvature for a parabolic spiral; also for the hyperbolic spiral.

Singular Points of Curves

59. It has been shown in Art. 36 that the first ratal coefficient of the equation of a curve represents the tangent of the angle of tangency; therefore, since the tangent of this angle is zero when the angle is zero, and infinite when the angle is 90°, it follows that the roots of the equation

$$\frac{dy}{dx} = 0,$$

will give the abscissas of all points of the curve at which the tangent line is parallel to the axis of abscissas; also that the roots of the equation

$$\frac{dy}{dx} = \infty,$$

will give the abscissas of all points of the curve at which the tangent line is perpendicular to the axis of abscissas (see Fig. 32 and Fig. 33).

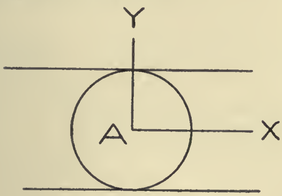


Fig. 32

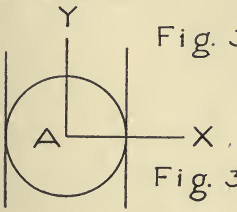


Fig. 33

Taking the equation of the circle

$$y = \pm (R^2 - x^2)^{1/2}$$

and passing to the rate,

$$\frac{dy}{dx} = \mp \frac{x}{(R^2 - x^2)^{1/2}}. \quad (1)$$

If (1) $= 0$, $x = 0$, but when $x = 0$, $y = \pm R$; therefore the circle has two tangents parallel to the axis of abscissas, (see Fig. 32). If (1) is infinite, $x = \pm R$ and $y = 0$; therefore the circle has two tangents perpendicular to its axis of abscissas (see Fig. 33).

Of the equation of the parabola,

$$y = \pm \sqrt{(2px)},$$

the rate is

$$\frac{dy}{dx} = \pm \frac{p}{\sqrt{(2px)}}. \quad (2)$$

If (2) $= 0$, both x and y are infinite; therefore the parabola has no tangent parallel to its axis of abscissas. If (2) is infinite, both x and y are equal to zero; therefore the parabola has a tangent (the axis of ordinates), perpendicular to its axis of abscissas at the origin of its coördinates, as shown in Fig. 34.

Fig. 34

Points of Inflection

60. Those points of a curve at which the curve changes its direction—that is, from being concave to its axis of abscissas it

becomes convex, or vice versa—are called points of inflection.

At such a point the angle of tangency and consequently its tangent must either change from increasing to decreasing, or from decreasing to increasing; therefore the rate of variation of the tangent of the angle of tangency at a point of inflection will be zero, real, or infinite; zero when the angle of tangency is zero, real between 0° and 90°, and infinite when 90°. Hence since $\frac{d^2y}{dx^2}$ represents the rate of variation of the tangent of the angle of tangency, every point of inflection will have for its abscissa some root of the equations:

$$\frac{d^2y}{dx^2}=0 \;(1),\; \frac{d^2y}{dx^2}>0 \;(2), \text{ and } \frac{d^2y}{dx^2}=\infty \;(3).$$

But it does not follow that every root of these equations will be the abscissa of a point of inflection; hence it is necessary to examine whether the value of x will give $\frac{d^2y}{dx^2}$ contrary signs (see Art. 37).

Let the equation of the curve be

$$y=a+b\,(x-c)^3. \tag{4}$$

Then passing to the rate twice,

$$\frac{dy}{dx}=3b\,(x-c)^2 \tag{5}$$

and

$$\frac{d^2y}{dx^2}=6b\,(x-c). \tag{6}$$

Making (6) equal to zero, then

$$x=c,$$

but when $x=c$, the first ratal coefficient is equal to zero also; therefore $y=a$ when $x=c$, there is a tangent line to the curve at the point whose coördinates are a and c, which is parallel to the axis of abscissas.

If b is positive, the second ratal coefficient will be zero for $x=c$, but negative when $x<c$ and positive when $x>c$; therefore there is an inflection of the curve at the point whose abscissa is $x=c$ (see Fig. 35).

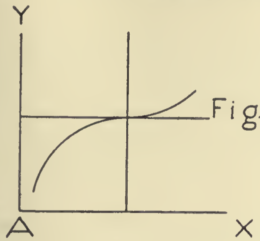


Fig. 35

If b is negative, the second ratal coefficient will be positive when $x<c$ and negative when $x>c$; therefore, at the point of the curve whose abscissa is $x=c$, there is an inflection of the curve, but opposite to the first (see Fig. 36).

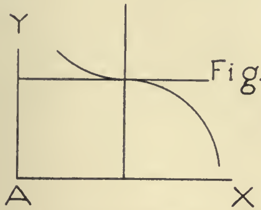


Fig. 36

In the first case the curve is first concave, then convex to the axis of X; in the second case it is first convex, then concave, as shown in the figures.

Let the equation of the curve be

$$y=a+b\,(x-c)^{3/5}.$$

Then, passing to the rate twice,

$$\frac{dy}{dx}=\frac{3b}{5\,(x-c)^{2/5}} \text{ and } \frac{d^2y}{dx^2}=\frac{6b}{25\,(x-c)^{7/5}}.$$

Making $x=c$, both expressions become infinite; therefore the first ratal coefficient, since $y=a$ when $x=c$, gives a tangent line to the curve at the point whose coördinates are a and c, which is perpendicular to the axis of X.

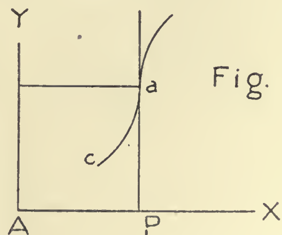


Fig. 37

If b is positive, the second ratal coefficient will be positive for all values of $x<c$, and negative for all values of $x>c$; hence, for all values of x less than c, which makes y positive, the curve will be convex to the axis of X, while for all values of x greater than c, it will be concave (see Fig. 37).

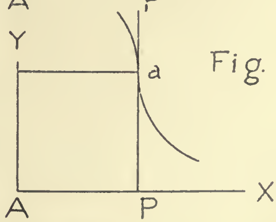


Fig. 38

If b is negative, the case will be the reverse, as shown in Fig. 38.

If $a=0$, P will be the point of inflection, and if $a=0$, also $c=0$, A will be the point of inflection.

Cusp Points

61. The point at which two branches of a curve terminate and have a common tangent is called a *cusp point.* When the cusp is formed by the union of two branches, one on either side of the tangent, it is called *a cusp of the first order,* and when both branches are on the same side of the tangent, it is called *a cusp of the second order.*

If $x=c$ be the abscissa of a cusp point, the values of x immediately preceding and following that of $x=c$, when substituted in the given equation, will give to y either two real or two imaginary values; if real, both will be greater or both less than that of the cusp point; furthermore, for a cusp point there will be a distinguishable term in the second ratal coefficient, either equal to zero or infinity.

Let the equation of the curve be

$$y=ax \pm b\,(x-c)^{5/2};$$

then, taking the rate twice gives

$$\frac{dy}{dx}=a \pm \frac{5}{2} b\,(x-c)^{3/2} \text{ and } \frac{d^2y}{dx^2}=\pm \frac{15}{4} b\,(x-c)^{1/2}.$$

Making the second ratal coefficient equal to zero, we have $x=c$; hence, since for a value of x less than c, y will have two imaginary values, and for a value of x greater than c, y will have two real values, there is a cusp at the point of the curve whose abscissa is $x=c$ (see Fig. 39).

When $x=c$, the first ratal coefficient equals a; hence the tangent of the angle of tangency at the cusp point is equal to a, and since $y=ac$ when $x=c$, the tangent line to the curve at the cusp passes through the origin of coördinates. Also for any value of x greater than c, $\frac{d^2y}{dx^2}$ will have two values, one positive and the other negative; consequently one branch of the curve is convex and the other concave to the axis of X; therefore a branch must lie on either side of the tangent line AN and the cusp is of the first order.

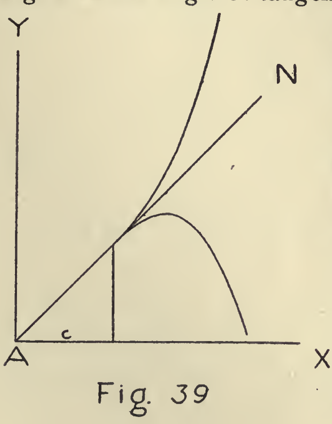


Fig. 39

The equation of the semi-cubical parabola is

$$y = \pm a x^{3/2},$$

the rates of which are

$$\frac{dy}{dx} = \pm \frac{3}{2} a x^{1/2} \text{ and } \frac{d^2y}{dx^2} = \pm \frac{3a}{4x^{1/2}}.$$

Making $\frac{d^2y}{dx^2} = \infty$, $x = 0$; then y has two imaginary values for $x < 0$ and two real values for $x > 0$; and, since $y = 0$ when $x = 0$, there is a cusp at the origin of the coördinates; but when $x = 0$, $\frac{dy}{dx} = 0$, hence the axis of abscissas is a tangent to both branches of the curve at the cusp (see Fig. 40).

Examination of the primitive equation shows that for every value of x greater than 0, y has two values, one positive and the other negative; therefore one branch of the curve, AC, lies above, and the other, AC', below the axis of X, and the cusp is of the first order. For any value of $x > 0$, $\frac{d^2y}{dx^2}$ has two values, one positive and the other negative; consequently, since y is negative for the branch AC', both branches are convex to the axis of X.

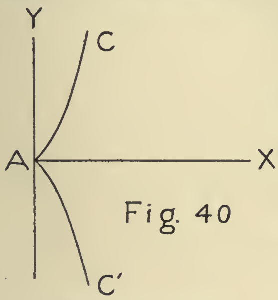


Fig. 40

Of the equation

$$y = a + b\,(x - c)^{2/3},$$

the rates are

$$\frac{dy}{dx} = \frac{2b}{3\,(x - c)^{1/3}}$$

and

$$\frac{d^2y}{dx^2} = -\frac{2b}{9\,(x - c)^{4/3}}.$$

Making the second ratal coefficient equal infinity. $x = c$; hence, since $y = a$ when $x = c$, and since y is greater than a either for $x < c$ or $x > c$, there is a cusp at the point of the curve whose coördinates are $x = c$, $y = a$.

When the first ratal coefficient is equal to infinity, $x = c$; hence a tangent line to the curve at the cusp point is perpendicular to the axis of abscissas (see Fig. 41).

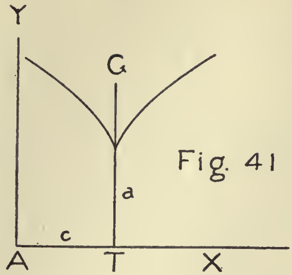


Fig. 41

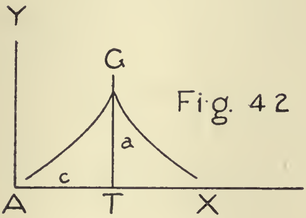


Fig. 42

For any value of x, either less or greater than c, the value of $\frac{d^2y}{dx^2}$ is negative; consequently both branches of the curve are concave to the axis of abscissas; also, since y has a value corresponding to either $x < c$ or $x > c$, a branch lies on either side of the tangent line TG.

If b is negative, then $\frac{d^2y}{dx^2}$ becomes positive for any value of x, either less or greater than c; therefore both branches of the curve are convex to the axis of abscissas (see Fig. 42).

Of the equation

$$y = 2x^2 \pm \frac{4}{5} x^{5/2},$$

the rates are
$$\frac{dy}{dx} = 4x \pm 2x^{3/2}$$

and
$$\frac{d^2y}{dx^2} = 4 \pm 3x^{1/2}.$$

Making the first ratal coefficient equal to zero, then $x = 0$ or 4; but when $x = 0$, $y = 0$, and when $x = 4$, $y = \frac{32}{5}$ or $\frac{288}{5}$; therefore the axis of abscissas is tangent to the curve at A, the origin. There is another tangent to the curve at E, parallel to the axis of X, and corresponding to an abscissa of 4 and an ordinate of $\frac{32}{5}$.

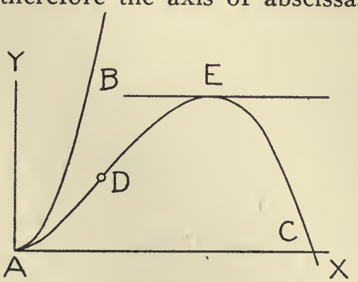


Fig. 43

If the positive sign be used (since $\frac{d^2y}{dx^2}$ will then be positive for any value of x), the left hand branch of the curve, AB, is convex to the axis of abscissas. But if the negative sign be used which corresponds with the right hand branch of the curve AEC, since $\frac{d^2y}{dx^2}$ will then be positive for $x < \frac{16}{9}$ and negative for $x > \frac{16}{9}$, the part AD, answering to $x = 0$ to $\frac{16}{9}$, will be convex and the part DEC concave; consequently there is an inflection at D.

In conclusion, since y has two imaginary values for $x < 0$, and two real values for $x > 0$, and since $y = 0$ when $x = 0$, the branches AB and AEC form a cusp of the second order at A, their origin.

Multiple Points

62. The points at which two or more branches of a curve intersect are called a *multiple point.*

At a multiple point it is therefore evident that there must be as many tangents to the curve as there are intersecting branches; hence $\frac{dy}{dx}$, which represents the tangent of the angle of tangency, will have as many values as there are different tangents.

Let the equation of the curve be

$$y = a \pm x\,(b^2 - x^2)^{1/2}; \qquad (1)$$

then, passing to the rate,

$$\frac{dy}{dx} = \pm \frac{b^2 - 2x^2}{(b^2 - x^2)^{1/2}}. \qquad (2)$$

An inspection of (1) shows that values of x greater than b make y imaginary, while for values of x less than b, y has two values; hence the curve has two branches which, since $y = a$

when $x=0$, intersect the axis of ordinates at a distance a from A, the origin (see Fig. 44).

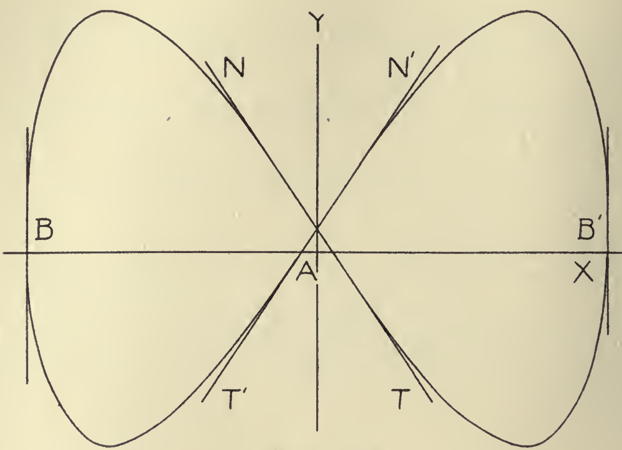


Fig. 44

When $x=b$ or $-b$, $\frac{dy}{dx}$ becomes infinite, consequently the tangents to the curve at B and B' are perpendicular to the axis of abscissas. When $x=0$, $\frac{dy}{dx}=\pm b$; therefore there are two tangents, TN and $T'N'$, to the curve passing through the multiple point.

Let the equation of the curve be

$$y=1\pm(1\mp\sqrt{x})\,(1\pm\sqrt{x})^{\frac{1}{2}}; \qquad (3)$$

then, passing to the rate,

$$\frac{dy}{dx}=-\frac{1\mp 3\sqrt{x}}{4\,(\sqrt{x}\pm x)^{\frac{1}{2}}}. \qquad (4)$$

From an examination of (3) we find for $x<0$ that y is imaginary; for $x=0$, $y=0$ or 2; for any value between 0 and 1, that y has four real values; for $x=1$, $y=1$; and for any value of x greater than 1, that y has only two real values.

Hence two branches of the curve must intersect each other at the point whose coördinates are $x = 1$ and $y = 1$ (see Fig. 45).

When $x = 1$, $\frac{dy}{dx}$ is infinite, $+\frac{1}{2}\sqrt{2}$, or $-\frac{1}{2}\sqrt{2}$, consequently the curve has a multiple point corresponding to the coördinates $x = 1$, $y = 1$, and at this point there are three

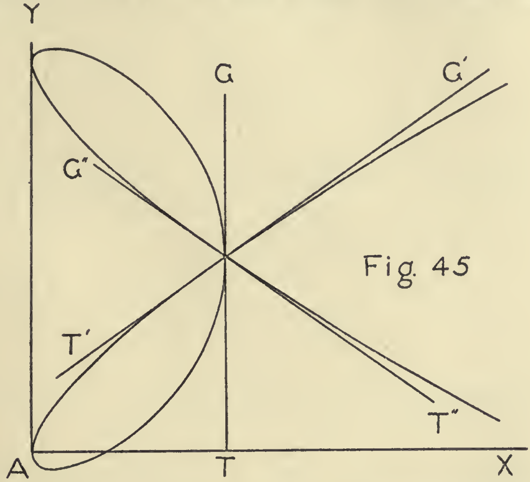


Fig. 45

tangent lines, TG, $T'G'$, $T''G''$: TG is perpendicular to the axis of abscissas, and $T'G'$ and $T''G''$ make angles therewith, whose respective tangents are $+\frac{1}{2}\sqrt{2}$ and $-\frac{1}{2}\sqrt{2}$.

Isolated Points

63. A point which is entirely detached from a curve, but whose coördinates satisfy the equation, is called an *isolated or conjugate point.*

Since a point entirely detached from a curve can have no tangent, it is evident that for an isolated point, the first ratal coefficient of the equation will be imaginary.

Let the equation be

$$y = \pm (x + a)\sqrt{x}; \qquad (1)$$

then, passing to the rate,

$$\frac{dy}{dx} = \pm \frac{3x + a}{2\sqrt{x}}. \qquad (2)$$

By examining (1) we find that $x=0$ makes $y=0$, and for any value of $x>0$, y has two real values, one positive and the other negative; therefore the curve passes through the origin A and has two branches, AC, AC', extending to the right (see Fig. 46).

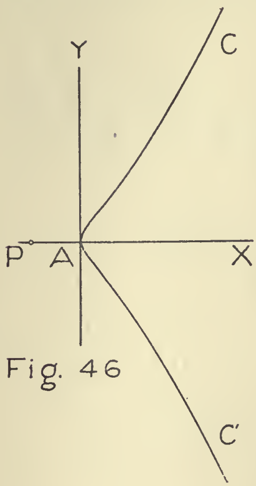


Fig. 46

Equation (1) is also satisfied by the coördinates $x=-a$ and $y=0$; but when $x=-a$, $\frac{dy}{dx}$ becomes imaginary; hence, the point P, whose abscissa is $x=-a$, being entirely detached from the curve, is an isolated point.

The rate of (2) is

$$\frac{d^2y}{dx^2}=\pm\frac{3x-a}{4x\sqrt{x}}.$$

Making this equal to zero gives $x=\frac{1}{3}a$; therefore there is an inflection of the curve at the point whose abscissa is $x=\frac{1}{3}a$.

Maxima and Minima

64. If a variable quantity increases until it attains a value greater than any immediately preceding or following it, such a value is called a maximum; and if it decreases until it attains a value less than any immediately preceding or following it, such a value is called a minimum.

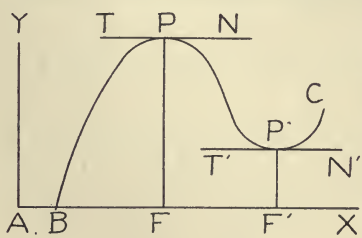


Fig. 47

Illustration. Let the points P and P' be so situated in the curve $BPP'C$ (see Fig. 47) that tangent lines, TN and $T'N'$, to the curve at P and P', shall be parallel to the axis of abscissas, AX; then, since it is obvious that the ordinate FP is greater than any immediately preceding or following it, and that the ordinate $F'P'$ is less than any immediately preceding or following it, FP is a maximum and $F'P'$ a minimum.

Therefore, if $y = f(x)$ is the equation of the curve, x representing any abscissa and y the corresponding ordinate, y is a maximum when it is equal to $F'P'$.

Now, for that point of a curve at which a tangent line is parallel to the axis of abscissas, since it then makes no angle with this axis, and since the first ratal coefficient of its equation represents the tangent of the angle of tangency (see Art. 36),

$$\frac{dy}{dx} = 0.$$

Hence *when a function is either a maximum or a minimum the first ratal coefficient is equal to zero.* For instance, if the function is of the form

$$y = x^2 - 2ax + b,$$

the rate is

$$\frac{dy}{dx} = 2x - 2a;$$

therefore y is either a maximum or a minimum when

$$2x - 2a = 0 \text{ or when } x = a.$$

It will be observed that the curve $BPP'C$ is concave to its axis of abscissas at the point P, and convex at the point P'; therefore, from Art. 37, since either ordinate FP or $F'P'$ is positive, the second ratal coefficient of the equation of the curve will be negative for the ordinate FP and positive for the ordinate $F'P'$. But it has been shown that FP is a maximum ordinate, and $F'P'$ a minimum ordinate; consequently, if the equation of the curve is $y = f(x)$, $\frac{d^2y}{dx^2}$ will be negative when y is a maximum and positive when y is a minimum.

Hence, to find the values of the variable of a function which will render the function a maximum or a minimum, also to distinguish the one from the other, we have the following rule.

Make the first ratal coefficient of the function equal to zero and find the values of the variable in this equation; then substitute these values in the second ratal coefficient of the function, and each value which gives a negative result will render the function a maximum, and each value which gives a positive result will render it a minimum.

It sometimes happens that a value of the variable, as $x=a$, found by making the first ratal coefficient of the function equal to zero, will reduce the second ratal coefficient to zero also. In this case, substitute $a \pm v$ for x in $\frac{d^2y}{dx^2}$, and if either $+v$ or $-v$ give a negative result for a small value of v, y will be a maximum; but if the result is positive, y will be a minimum. If one sign gives a negative result and the other a positive result, it is clear y will be neither a maximum nor a minimum; such a result simply indicates that the curve represented by the proposed equation, has an inflection at the point corresponding to the abscissa $x=a$ (see Art. 60).

For illustration, take the equation

$$y = x^4 - 4x^3 + 16x + 13;$$

then, passing to the rate twice,

$$\frac{dy}{dx} = 4x^3 - 12x^2 + 16, \tag{1}$$

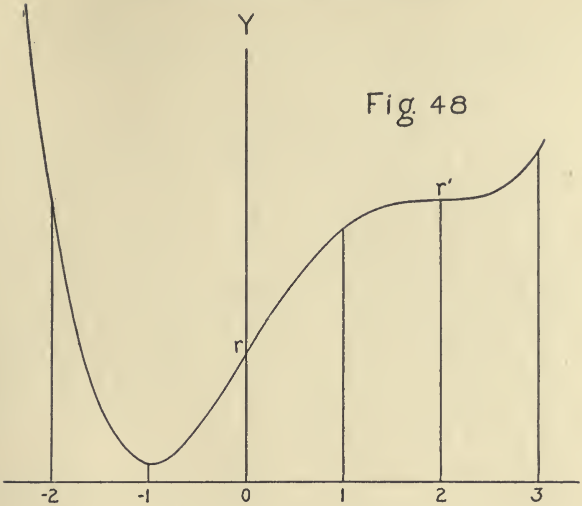
and

$$\frac{d^2y}{dx^2} = 12x^2 - 24x. \tag{2}$$

If $(1)=0$, then $x=-1$ or $x=2$.

Substituting these values of x in (2), then for $x=-1$, $\frac{d^2y}{dx^2}=36$ and for $x=2$, $\frac{d^2y}{dx^2}=0$.

Since $x=2$ reduces the second ratal coefficient to zero, by substituting $2 \pm v$ for x in (2) the result is $12v\,(v \pm 2)$, which, for a small value of v, is negative for the minus sign and positive for the plus sign, which shows that there is an inflection of the curve at r', corresponding to the abscissa $x=2$; hence this value of x makes y neither a maximum nor a minimum (see Fig. 48). There is also an inflection at r, but a minimum for y answering to $x=-1$.



Let the equation be

$$y = (x^3 - 12x)^{\frac{1}{2}};$$

then $$\frac{dy}{dx} = \frac{\frac{1}{2}(3x^2 - 12)}{(x^3 - 12x)^{\frac{1}{2}}} = 0, \text{ whence } x = \pm 2.$$

Taking the rate of (3) and reducing, regarding $3x^2 - 12 = 0$, the result is

$$\frac{d^2y}{dx^2} = \frac{3x}{(x^3 - 12x)^{\frac{1}{2}}},$$

which, for $x = -2$, equals $-\frac{3}{2}$, and for $x = 2$, equals $-\frac{3}{2}\sqrt{-1}$; therefore y is a maximum when $x = -2$ and a minimum when $x = +2$.

If $$y^2 = v = x^3 - 12x,$$

then $$\frac{dv}{dx} = 3x^2 - 12;$$

whence $$x = \pm 2,$$

the same values of x as before.

Hence *when the expression containing the variable is under a radical, the radical may be omitted.*

65. In case the equation is of the form

$$f(x, y) = 0, \tag{1}$$

in which y is a function of x, pass to the rate and find from $\frac{dy}{dx} = 0$ the value of x in terms of y, also of y in terms of x. Substitute the value of x in terms of y in (1) and therefrom determine the value of y, and with this value of y find that of x. Next determine the second ratal coefficient and substitute therein the values of x and y found from (1). And if the result is negative, y will be a maximum; but if positive, a minimum. Should these values reduce the second ratal coefficient to zero, proceed as in article 64.

EXAMPLE

$$y^2 + 2axy - x^2 - b^2 = 0. \tag{2}$$

Passing to the rate,

$$2ydy + 2axdy + 2aydx - 2xdx = 0,$$

or

$$\frac{dy}{dx} = \frac{x - ay}{ax + y}, \tag{3}$$

whence, by making it equal to zero,

$$x = ay \text{ or } y = \frac{x}{a}.$$

Substituting ay for x in (2), we find

$$y = \frac{b}{(1 + a^2)^{1/2}};$$

therefore

$$x = \frac{ab}{(1 + a^2)^{1/2}}.$$

Taking the rate of (3), regarding $\frac{dy}{dx} = 0$, also $x - ay = 0$,

then

$$\frac{d^2y}{dx^2} = \frac{ax + y}{(ax + y)^2} = \frac{1}{ax + y},$$

or, substituting for x and y their values and reducing,

$$\frac{d^2y}{dx^2} = \frac{1}{b(1 + a^2)^{1/2}}.$$

Hence y is a minimum when $x = \dfrac{ab}{(1+a^2)^{1/2}}$.

The value of $x = a$ found from $\dfrac{dy}{dx} = 0$ will sometimes make the second ratal coefficient infinite. In such a case, substitute $a \pm v$ (v being a small quantity) for x in $\dfrac{d^2y}{dx^2}$. Then if the result for both signs of v is negative, y will be a maximum for $x = a$, and if positive, a minimum; but if the result for one sign is negative and for the other sign positive, y will be neither a maximum nor a minimum.

Let the function be

$$y = b - (x - a)^{4/3}.$$

Passing to the rate twice,

$$\frac{dy}{dx} = -\frac{4}{3}(x-a)^{1/3}, \tag{4}$$

and for $x = a$, $$\frac{d^2y}{dx^2} = -\frac{4}{9(x-a)^{2/3}} = \infty. \tag{5}$$

If (4) $= 0$, $x = a$; but when $x = a$, (5) becomes infinite; therefore, by substituting $a \pm v$ for x in (5),

$$\frac{d^2y}{dx^2} = -\frac{4}{9(\pm v)^{2/3}},$$

which is negative for either plus or minus v; therefore y is a maximum when $x = a$.

66. When the curve represented by a given equation forms a cusp of the first order, and a tangent line to the curve at the cusp point makes an angle of ninety degrees with the axis of abscissas, it is evident, as may be seen in the figures 41 and 42, that the ordinate of the cusp point will be a maximum or minimum, according as both branches of the curve are concave or convex to the axis of abscissas; but when the angle of tangency is 90°, the first ratal coefficient of the equation is infinite. Therefore, to find the value of the variable which will render the function a maximum or a minimum in such a case, a solution is required of

$$\frac{dy}{dx} = \infty.$$

If the value of x thus found renders the second ratal coefficient infinite, proceed as has been previously explained.

For an example, take

$$y = a - 9(x - c)^{2/3};$$

then, passing to the rate twice,

$$\frac{dy}{dx} = -\frac{6}{(x-c)^{1/3}} = \infty \qquad (1)$$

and

$$\frac{d^2y}{dx^2} = \frac{2}{(x-c)^{4/3}} = \infty. \qquad (2)$$

(1) is satisfied when $x = c$, and by substituting $c + v$ in (2), it becomes

$$\frac{d^2y}{dx^2} = \frac{2}{v^{4/3}};$$

therefore y is a minimum when $x = c$.

EXAMPLES

Determine the values of the variable that will make the following maxima or minima.

1. $y = x^4 - 8x^3 + 22x^2 - 24x.$

2. $y = b - (x - a)^{4/3}.$

3. $y = 4 \pm (3x^2 - 12x + 9)^{2/3}.$

4. Divide a quantity, a, into two such parts, that the mth power of one part multiplied by the nth power of the other part shall be a maximum.

5. Determine the minimum hypotenuse of a right-angled triangle containing an inscribed rectangle whose sides are as a to b.

6. Determine the length of the axis of the largest parabola that can be cut from a right cone, the length of whose side is s.

7. The perpendiculars of two right-angled triangles are a and b, the sum of their bases c, and the sum of their hypotenuses a minimum. What is the base of each?

8. If the solidity of a cylinder is 2π and its surface a minimum, what is its diameter?

9. What is the height of the largest cylinder which can be inscribed in a cone whose altitude is a?

10. What is the altitude of a maximum rectangle inscribed in a triangle whose base is b and perpendicular height a?

11. What is the altitude of the largest cylinder that can be cut from a paraboloid whose axis is a?

67. It has been shown that the value of a single variable which will render its function a maximum or a minimum, is found by making the first ratal coefficient of the function equal to zero; hence it is evident that the value of each variable of a function of two or more variables is also to be found by making the first partial ratal coefficient of the function, relative to that variable, equal to zero: that is, if $u=f(x, y)$, the value of x which will render the function a maximum or a minimum, is found from

$$\frac{du}{dx}=0,$$

and of y from

$$\frac{du}{dy}=0,$$

whence all the values of x and y can be found, which will render u a maximum or minimum.

It has also been shown that the second ratal coefficient of a function of a single variable is negative when the function is a maximum and positive when it is a minimum; for like reasons, the values of x and y, found from the first partial rates of $u=f(x, y)$, when substituted in the second partial rates, must give each a negative value when u is a maximum and a positive value when u is a minimum.

The second partial rates of $u=f(x, y)$ are

$$\frac{d^2u}{dx^2}, \frac{d^2u}{dy^2}, \frac{d^2u}{dxdy}, \text{ and } \frac{d^2u}{dydx},$$

or, since the last two expressions are equal (see Art. 21), only the following need be used, namely

$$\frac{d^2u}{dx^2}, \frac{d^2u}{dy^2}, \text{ and } \frac{d^2u}{dxdy},$$

each of which must be negative when u is a maximum and positive when u is a minimum.

The process is similar when there are three or more independent variables.

Let $$u = ax^3y^2 - x^4y^2 - x^3y^3,$$

the partial rates of which are

$$\frac{du}{dx} = 3ax^2y^2 - 4x^3y^2 - 3x^2y^3$$

and $$\frac{du}{dy} = 2ax^3y - 2x^4y - 3x^3y^2.$$

Making these equal to zero, it will be found that

$$x = \frac{1}{2}a \text{ and } y = \frac{1}{3}a.$$

The second partial rates are

$$\frac{d^2u}{dx^2} = 6axy^2 - 12x^2y^2 - 6xy^3,$$

$$\frac{d^2u}{dy^2} = 2ax^3 - 2x^4 - 6x^3y$$

and $$\frac{d^2u}{dxdy} = 6ax^2y - 8x^3y - 9x^2y^2.$$

Substituting in these the values of x and y, the results are

$$\frac{d^2u}{dx^2} = -\frac{a^4}{9}, \frac{d^2u}{dy^2} = -\frac{a^4}{8} \text{ and } \frac{d^2u}{dxdy} = -\frac{a^4}{6};$$

therefore, when $x = \frac{a}{2}$ and $y = \frac{a}{3}$, u is a maximum and equal to $\frac{a^6}{432}$.

EXAMPLES

The volume of a rectangular solid is s. What is the length of each side when its surface is a minimum?

Let x, y, and $\frac{s}{xy}$ represent the lengths of the sides, and u its surface.

Then $$u = 2xy + \frac{2s}{x} + \frac{2s}{y}.$$

The first partial rates of this are

$$\frac{du}{dx} = 2y - \frac{2s}{x^2} \text{ and } \frac{du}{dy} = 2x - \frac{2s}{y^2};$$

whence, by making them equal to zero, it will be found that

$$x = s^{1/3} \text{ and } y = s^{1/3}.$$

The second partial rates are

$$\frac{d^2u}{dx^2} = \frac{4s}{x^3}, \quad \frac{d^2u}{dy^2} = \frac{4s}{y^3} \text{ and } \frac{d^2u}{dxdy} = 2.$$

Since each of these are positive, u is a minimum when each side is equal to $s^{1/3}$.

The semi-diameter of a sphere is r. What are the lengths of the sides of the greatest rectangular parallelopipedon that can be cut from it?

PART TWO
THE INVERSE METHOD

PART TWO

THE INVERSE METHOD

DEFINITIONS AND ILLUSTRATIONS

68. In Part One the function is given to find the rate; herein the rate of the function is given to find the function.

69. The method of passing from the rate to the function—that is, the process of restoring the function of which the rate is given—is called *integration*, and the restored function is called the *integral* of that rate.

The integral of a given ratal expression is indicated by the character $\int$ placed before it, as

$$\int (2axdx + bdx),$$

showing that the integral is required.

70. There can be only one rate of a given function, but there may be more than one function answering to a given rate. This is obvious, since x^2 and $x^2 + a$ have the same rate, viz., $2xdx$. Therefore, in integrating, a constant term must be added to the integral. This term is usually represented by C; thus the integral of

$$du = 2axdx \text{ is } u = ax + C.$$

C is called an arbitrary constant, and the integral before the value of C is known is called an incomplete integral. In the solution of a real problem, however, the value of C may be determined from the known conditions of the problem and consequently a complete integral obtained.

For illustration take the ratal equation of the straight line,

$$dy = adx,$$

whence $$y = ax + C.$$

If the straight line passes through the origin of the coördinates, then $y = 0$ when $x = 0$; hence $C = 0$, and the complete or true integral is

$$y = ax.$$

But if the straight line cuts the axis of ordinates at a distance from the origin equal to b, then for $x=0$, $y=b$, consequently $C=b$, and the true integral is

$$y=ax+b.$$

71. Of the triangle ABC, let AC be represented by x, BC by $2ax$, CD by dx, and the area of ABC by A; then

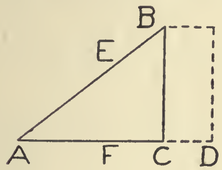


Fig. 49

$$dA=2axdx,$$

whence $$A=ax^2+C.$$

But when $x=0$, $A=0$, consequently $C=0$; therefore the true integral is $A=ax^2$. (1)

It will be observed that the same is true for the triangle AEF when $x=AF$, and $A'=$ the area of AEF—that is

$$A'=ax^2. \qquad (2)$$

Now if $x=n$ in (1) and $x=m$ in (2), representing the area of $EBFC$ by A'', then

$$A''=A-A'=an^2-am^2.$$

This process is termed integrating between limits. In the present case, the integral of $2xdx$ is taken between the limits of $x=m$ and $x=n$, m being called the inferior limit of x and n the superior limit. The sign of this method is placed before the given rate; thus $\int_m^n Xdx$, X being a function of x. If $m=0$, then the sign becomes $\int_0^n$.

Simple Algebraic Rates

72. According to the rules under Art. 10, if

$$u=\frac{ax^{n+1}}{n+1}, du=ax^n dx;$$

therefore, it is seen that the function corresponding to the rate

$$ax^n dx \text{ is, by Art. 70, } \frac{ax^{n+1}}{n+1}+C:$$

that is $$\int ax^n dx=\frac{ax^{n+1}}{n+1}+C.$$

Hence the following rule:

The integral of a monomial rate is equal to the constant factor into the variable with its exponent increased by unity, divided by the exponent thus increased, plus a constant term.

This rule is applicable whether n is positive or negative, a whole number or a fraction, except when $n = -1$, for then

$$\frac{ax^{n+1}}{n+1} = \frac{ax^{1-1}}{1-1} = \frac{ax^0}{0} = \frac{a}{0} = \infty$$

But when $n = -1$,

$$ax^n dx = ax^{-1}dx = \frac{adx}{x},$$

which is the rate of $\log x$, by Art. 27,

therefore $$\int \frac{adx}{x} = a \log x + C.$$

Hence *the integral of a fractional rate whose numerator is the rate of the denominator multiplied by a constant, is equal to the constant into the Naperian logarithm of the denominator, plus a constant term.*

Since a constant quantity retains the same value throughout the same investigation, it can be placed outside the sign of integration, as $a\int x^n dx$.

73. Since the rate of a function composed of the sum or difference of any number of terms containing the same independent variable is the corresponding sum or difference of their rates taken separately (see Art. 11), it follows that the integral of a ratal expression composed of the sum or difference of several terms is equal to the corresponding sum or difference of their respective integrals; thus

$$\int (ax^2dx + bdx - nx^{n-1}dx) = a\int x^2dx + b\int dx - n\int x^{n-1}dx = \frac{1}{3}ax^3 + bx - x^n + C.$$

From this it is evident that a polynomial of the form

$$du = (a \pm bx \pm cx^2 \pm \text{etc.})^n \, dx,$$

in which n is a positive whole number, can be integrated by raising the quantity within the parenthesis to the nth power, multiplying through by dx, then integrating each term separately.

Let $$du = (a + bx)^3\, dx;$$
then $$du = a^3dx + 3a^2bxdx + 3ab^2x^2dx + b^3x^3dx,$$
whence $$u = \int (a^3dx + 3a^2bxdx + 3ab^2x^2dx + b^3x^3dx) =$$
$$a^3x + \frac{3}{2}a^2bx^2 + ab^2x^3 + \frac{1}{4}b^3x^4 + C.$$

When the rate is of the form
$$du = (x^2 + ax + b)^n\,(2xdx + adx),$$
in which n is an integer or fraction, positive or negative, and when the quantity within the last parenthesis is the rate of that within the first; then
$$u = \int (x^2 + ax + b)^n\,(2xdx + adx) =$$
$$\frac{1}{n+1}(x^2 + ax + b)^{n+1} + C.$$

This case is substantially the same as that of a monomial rate (see Art. 72) and is similarly inapplicable under the same condition: viz., when the exponent $n = -1$, for then
$$\frac{1}{n+1}(x^2 + ax + b)^{n+1} = \frac{1}{1-1}(x^2 + ax + b)^{1-1} =$$
$$\frac{(x^2 + ax + b)^0}{1-1} = \frac{1}{0} = \infty;$$
but when $n = -1$,
$$(x^2 + ax + b)^n\,(2xdx + adx) =$$
$$(x^2 + ax + b)^{-1}\,(2xdx + adx) = \frac{2xdx + adx}{x^2 + ax + b},$$
in which the numerator is the rate of the denominator; therefore
$$u = \int (x^2 + ax + b)^{-1}\,(2xdx + adx) = \int \frac{2xdx + adx}{x^2 + ax + b} =$$
$$\log\,(x^2 + ax + b) + C.$$

74. To determine the integral of a binomial rate of the form
$$du = (a + bx^n)^m\, x^{n-1}dx:$$
that is, one in which the exponent of the variable without the parenthesis is less by unity than that of the variable within.

Assume $$a + bx^n = y,$$
and taking the rate $$nbx^{n-1}dx = dy$$
or $$x^{n-1}dx = \frac{dy}{nb};$$
hence $$du = y^m \frac{dy}{nb} = \frac{y^m dy}{nb};$$
therefore by Art. 72, $$u = \frac{y^{m+1}}{nb\,(m+1)} + C;$$
or, substituting for y its value,
$$u = \frac{(a + bx^n)^{m+1}}{nb\,(m+1)} + C.$$

Hence *the integral of a binomial rate in which the exponent of the variable without the parenthesis is one less than that within, is equal to the binomial factor with its exponent increased by unity, divided by the exponent thus increased into the product of the exponent and coefficient of the variable within, with a constant term added to the result.*

If the rate is
$$du = \frac{(a + bnx^{n-1})\,dx}{2\,(ax + bx^n)^{1/2}},$$
or $$du = \frac{1}{2}\,(ax + bx^n)^{-1/2}\,(a + bnx^{n-1})dx,$$
it will be seen that the quantity within the last parenthesis is the rate of that within the first; therefore, by Art. 73,
$$u = \int \frac{1}{2}\,(ax + bx^n)^{-1/2}\,(a + bnx^{n-1})\,dx = (ax + bx^n)^{1/2}.$$

If the rate is
$$du = \frac{adx}{b \pm cx},$$
by making $$b \pm cx = y,$$
then $$\pm\, cdx = dy, \text{ or } dx = \frac{\pm\, dy}{c};$$
therefore $$du = \frac{\pm\, ady}{cy};$$

consequently, by Art. 72,

$$u = \pm \frac{a}{c} \log y + C,$$

or, substituting for y its value,

$$u = \pm \frac{a}{c} \log (b \pm cx) + C.$$

EXAMPLES

1. $du = \dfrac{ax^3 dx}{2}$

2. $du = (x^2 + ax)^2 (2xdx + adx)$

3. $du = (1 + ax)^{-3} 2xdx.$

4. $du = \dfrac{axdx}{(x^2 + a^2)}$

5. $du = (a + bx^2)^{\frac{1}{2}} mxdx + \dfrac{ndx}{c + x}$

Simple Circular Rates

75. Referring to Art. 31, it will be seen that

$$u = \int \cos xdx = \sin x$$

$$u = \int \frac{dx}{\cos^2 x} = \tan x$$

$$u = \int - \sin xdx = \cos x$$

$$u = \int - \frac{dx}{\sin^2 x} = \cot x$$

$$u = \int \sin xdx = \text{vers } x$$

$$u = \int \frac{\tan xdx}{\cos x} = \sec x, \text{ etc.}$$

Also in (3) of Art. 31, it is shown that the rate of

$$\sin x^n = n \cos x^{n-1} dx—$$

that is,
$$\int n \cos x^{n-1} dx = \sin x^n; \qquad (1)$$

hence it is clear that

$$\int - n \sin x^{n-1} dx = \cos x^n. \qquad (2)$$

If $n = 1$, (1) and 2 become

$$u = \sin x + C \text{ and } u = \cos x + C.$$

EXAMPLES

1. $\int \sin \frac{1}{x} \cdot \frac{dx}{x^2}$

2. $\int - \frac{2xdx}{\cos^2 (1 - x^2)}$

76. It is shown in Art. 33, making $R = 1$ and omitting the constant C, that

1. $x = \int \frac{du}{(1 - u^2)^{\frac{1}{2}}} = \sin^{-1} u$

2. $x = \int - \frac{du}{(1 - u^2)^{\frac{1}{2}}} = \cos^{-1} u$

3. $x = \int \frac{du}{(2u - u^2)^{\frac{1}{2}}} = \text{vers}^{-1} u$

4. $x = \int \frac{du}{1 + u^2} = \tan^{-1} u$

Let $$dx = \frac{du}{(a^2 - u^2)^{\frac{1}{2}}}, \qquad (1)$$

and assume $$u = av;$$

then $du = adv$ and $(a^2 - u^2)^{\frac{1}{2}} = a(1 - v^2)^{\frac{1}{2}}$.

Substituting these values in (1) gives

$$dx = \frac{dv}{(1 - v^2)^{\frac{1}{2}}};$$

hence [see (1)], $x = \int \frac{dv}{(1 - v^2)^{\frac{1}{2}}} = \sin^{-1} v$

or, since $\frac{dv}{(1 - v^2)^{\frac{1}{2}}} = \frac{du}{(a^2 - u^2)^{\frac{1}{2}}}$ and $v = \frac{u}{a}$,

$$x = \int \frac{du}{(a^2 - u^2)^{\frac{1}{2}}} = \sin^{-1} \frac{u}{a}.$$

Let $$dx = \frac{du}{(2au - u^2)^{\frac{1}{2}}},$$

and assume $$u = av;$$

then

$$du = adv \quad \text{and} \quad \frac{du}{(2au - u^2)^{\frac{1}{2}}} = \frac{adv}{a(2v - v^2)^{\frac{1}{2}}} = \frac{dv}{(2v - v^2)^{\frac{1}{2}}};$$

therefore [see (3)], $x = \int \frac{du}{(2v - v^2)} = \text{vers}^{-1} v =$

$$\int \frac{du}{(2au - u^2)^{\frac{1}{2}}} = \text{vers}^{-1} \frac{u}{a}.$$

Let $$dx = \frac{du}{a^2 + u^2},$$

and assume $u = av$; then $du = adv$

and $$\frac{du}{a^2 + u^2} = \frac{adv}{a^2(1 + v^2)} = \frac{dv}{a\,(1 + v^2)};$$

therefore

$$x = \frac{1}{a}\int \frac{dv}{1 + v^2} = \frac{1}{a} \tan^{-1} v = \int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a}.$$

EXAMPLES

1. $dx = -\frac{du}{(c - u^2)^{\frac{1}{2}}}$

2. $dx = \frac{du}{(4u - 2u^2)^{\frac{1}{2}}}$

3. $dx = -\frac{du}{5 + u^2}$

4. $dx = -\frac{du}{(1 - u^2)^{\frac{1}{2}}} + \frac{du}{(2u - u^2)^{\frac{1}{2}}}$

Integration by Series

77. Any expression of the form

$$du = Xdx,$$

in which X is such a function of x that it can be developed into a series of the powers of x, may be integrated in the following manner. Supposing the development to be

$$X = Ax^a + Bx^b + Cx^c + \text{etc.},$$

then multiplying by dx and integrating each term separately give

$$u = \int X dx = \frac{A}{a+1} x^{a+1} + \frac{B}{b+1} x^{b+1} + \frac{C}{c+1} x^{c+1} + \text{etc.}$$

This method is often the best, if not the only course to pursue, for when the series are rapidly converging, an approximate value of the integral may be readily determined.

Let $$du = \frac{dx}{a+x}.$$

Then, developing by the binomial theorem,

$$\frac{1}{a+x} = (a+x)^{-1} = \frac{1}{a} - \frac{x}{a^2} + \frac{x^2}{a^3} - \frac{x^3}{a^4} + \text{etc.};$$

multiplying by dx,

$$\frac{dx}{a+x} = \frac{dx}{a} - \frac{xdx}{a^2} + \frac{x^2dx}{a^3} - \frac{x^3dx}{a^4} + \text{etc.},$$

and integrating, the result is

$$u = \int \frac{dx}{a+x} = \left(\frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \text{etc.}\right) + C.$$

It has been shown in Art. 72 that

$$u = \int \frac{dx}{a+x} = \log\,(a+x) + C;$$

therefore

$$u = \log\,(a+x) = \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \text{etc.}—$$

a result, when $a = 1$, the same as found in Art. 29.

Let $$du = \frac{dx}{1+x^2} = (1+x^2)^{-1}\,dx.$$

Developing,

$$(1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \text{etc.}$$

Multiplying by dx and integrating,

$$u = \int (1+x^2)\,dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \text{etc.}$$

It has been shown in Art. 76 that

$$\int \frac{dx}{1+x^2} = \tan^{-1} x;$$

therefore $u = \int (1+x^2)^{-1}\, dx = \tan^{-1} x =$

$$\left(\frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \text{etc.}\right) + C.$$

When $x = 0$, the arc, and consequently C, equals 0; therefore

$$u = \tan^{-1} x = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \text{etc.}$$

Let $du = \dfrac{dx}{(1-x^2)^{1/2}} = (1-x^2)^{-1/2}\, dx.$

Developing and integrating, the result is

$$u = \int \frac{dx}{(1-x^2)^{1/2}} =$$

$$\left(\frac{x}{1} + \frac{x^3}{2\cdot 3} + \frac{3x^5}{2\cdot 4\cdot 5} + \frac{3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7} + \text{etc.}\right) + C.$$

Referring to Art. 76, it is found that

$$\int \frac{dx}{(1-x^2)^{1/2}} = \sin^{-1} x;$$

therefore $u = \sin^{-1} x =$

$$\left(\frac{x}{1} + \frac{x^3}{2\cdot 3} + \frac{3x^5}{2\cdot 4\cdot 5} + \frac{3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7} + \text{etc.}\right) + C.$$

Let $du = \dfrac{dx}{(x-x^2)^{1/2}}.$ (1)

Assuming $x = v^2$, then $dx = 2vdv$

and $\dfrac{dx}{(x-x^2)^{1/2}} = \dfrac{2dv}{(1-v^2)^{1/2}} = 2\,(1-v^2)^{-1/2}\, dv.$

Developing $2\,(1-v^2)^{-1/2}$, multiplying by dv, and integrating give

$$\int 2\,(1-v^2)^{-\frac{1}{2}}\,dv =$$
$$2\left(\frac{v}{1}+\frac{v^3}{2\cdot 3}+\frac{3v^5}{2\cdot 4\cdot 5}+\frac{3\cdot 5v^7}{2\cdot 4\cdot 6\cdot 7}+\text{etc.}\right)+C;$$

but $\int 2\,(1-v^2)^{-\frac{1}{2}}\,dv = 2\sin^{-1} v$; therefore, substituting for v its value, $x^{\frac{1}{2}}$, the result is

$$u=\int\frac{dx}{(x-x^2)^{\frac{1}{2}}}=2\sin^{-1}x^{\frac{1}{2}}.$$

Putting (1) under the form $\dfrac{2dx}{(2\cdot 2x-4x^2)^{\frac{1}{2}}}$ and assuming $2x=v$, then

$$du=\frac{2dx}{(2\cdot 2x-4x^2)^{\frac{1}{2}}}=\frac{dv}{(2v-v^2)^{\frac{1}{2}}};$$

but $$u=\int\frac{dv}{(2v-v^2)^{\frac{1}{2}}}=\text{vers}^{-1}\,v;$$

therefore $$u=\int\frac{2dx}{(2\cdot 2x-4x^2)^{\frac{1}{2}}}=\text{vers}^{-1}\,2x.$$

EXAMPLES

1. $du=(1+x^2)^{\frac{1}{2}}\,dx$
2. $du=(2ax-x^2)^{\frac{1}{2}}\,dx$
3. $du=(a+x)^2\,dx$
4. $du=(x^2-1)^{-\frac{1}{2}}\,dx$

Binomial Rates

78. If the rate is of the form

$$du=(a+bx^{-n})^r\,x^m dx,$$

assume $x=v^{-1}$, then $dx=-v^{-2}dv$ and $x^m=v^{-m}$; therefore

$$du=-(a+bv^n)^r\,v^{-m-2}dv,$$

in which the exponent of v within the parenthesis is positive.

If the rate is of the form

$$du=(ax^s+bx^n)^r\,x^m dx,$$

it can be written thus, s being less than n:

$$du=(a+bx^{n-s})^r\,x^{m+rs}dx,$$

in which only one term within the parenthesis contains the variable.

Finally, if the rate is of the form

$$du = (a + bx^n)^r x^m dx,$$

in which m and n are fractional, by substituting for x another variable having an exponent equal to the least common multiple of the denominators of m and n, a new binomial rate can be found in which the exponents of the variable will be whole numbers. Thus, if in the rate

$$du = (a + bx^{1/2})^r x^{1/3} dx,$$

v^6 be substituted for x, then, since $dx = 6v^5 dv$,

$$du = 6\,(a + bv^3)^r v^7 dv.$$

Hence any binomial rate can be reduced to one of the form

$$du = (a + bx^n)^r x^m dx, \qquad (1)$$

in which the exponents m and n are whole numbers and n is positive.

When r, the exponent of the parenthesis, is a positive whole number, (1) can be integrated as shown in Art. 73; also, when $m = n - 1$, as shown in Art. 74.

Assuming $$a + bx^n = v$$

in (1), then $$(a + bx^n)^r = v^r, \qquad (2)$$

whence $$x^n = \frac{v - a}{b}$$

and $$x^{m+1} = \left(\frac{v - a}{b}\right)^{(m+1)/n};$$

hence, by passing to the rate and dividing by $m + 1$,

$$x^m dx = \frac{1}{bn}\left(\frac{v - a}{b}\right)^{(m+1)/n-1} dv.$$

Multiplying this by 2 gives

$$(a + bx^n)^r x^m dx = \frac{1}{bn}\left(\frac{v - a}{b}\right)^{(m+1)/n-1} v^r dv; \qquad (3)$$

hence $$du = \frac{1}{bn}\left(\frac{v - a}{b}\right)^{(m+1)/n-1} v^r dv \qquad (4)$$

which can be integrated when $\frac{m+1}{n}$ is a positive whole number, or when $m + 1 = n^2$ (see Art. 72). If $\frac{m+1}{n}$ is negative, see formula D, Art. 80.

Let $$du = (a + bx^2)^{\frac{1}{2}} x^5 dx$$

in which $m = 5$, $n = 2$, $\frac{m+1}{n} = 3$, and $r = \frac{1}{2}$; then by substituting these values in (4), the following is obtained:

$$du = \frac{1}{2b} \left(\frac{v - a}{b}\right)^2 v^{\frac{1}{2}} dv = \frac{1}{2b^3} (v^{5/2} - 2av^{3/2} + a^2 v^{\frac{1}{2}})\, dv.$$

Then, by integrating and reducing,

$$u = \frac{1}{b^3} \left(\frac{v^3}{7} - \frac{2av^2}{5} + \frac{a^2 v}{3}\right) v^{\frac{1}{2}} + C;$$

therefore, since $$v = (a + bx^2),$$

$$u = \frac{1}{b^3} \left\{\frac{(a + bx^2)^3}{7} - \frac{2a\,(a + bx^2)^2}{5} + \right.$$

$$\left. \frac{a^2\,(a + bx^2)}{3}\right\} (a + bx^2)^{\frac{1}{2}} + C.$$

If $\frac{m+1}{n}$ is not a whole number, (1) may be written thus:

$$du = \{x^n\,(ax^{-n} + b)\}^r\, x^m dx = (ax^{-n} + b)^r\, x^{m+nr} dx.$$

By substituting in the right hand member of (3) $m + nr$ for m, $-n$ for n, a for b, and b for a, then

$$du = (b + ax^{-n})^r\, x^{m+nr} dx =$$

$$\frac{1}{-an} \left(\frac{v - b}{a}\right)^{(m+nr+1)/-n-1} v^r dv, \qquad (5)$$

which can be integrated by Art. 72 when $\frac{m + nr + 1}{-n}$ is a positive whole number; if negative, by formula D of Art. 80.

79. Referring to Art. 9, it is found that

$$d\,(vz) = vdz + zdv,$$

whence, by integrating,

$$vz = \int vdz + \int zdv;$$

hence $$\int vdz = vz - \int zdv, \tag{1}$$

in which it is seen that the integral of vdz depends upon that of zdv.

Resuming (1) of the last article,

$$du = (a + bx^n)^r x^m dx, \tag{2}$$

and assuming $$z = (a + bx^n)^s,$$

in which the exponent s may have such a value assigned to it as may be found most convenient; then, by passing to the rate,

$$dz = bns\,(a + bx^n)^{s-1} x^{n-1} dx. \tag{3}$$

Again, assuming $vdz = (a + bx^n)^r x^m dx$, and dividing it by (3),

$$v = \frac{(a + bx^n)^{r-s+1} x^{m-n+1}}{bns}$$

and, passing to the rate,

$$du = \frac{\left[\begin{array}{l}\{(m - n + 1)\,(a + bx^n)^{r-s+1} x^{m-n} + \\ bn\,(r - s + 1)\,(a + bx^n)^{r-s} x^m\}\,dx\end{array}\right]}{bns} =$$

$$\left\{\frac{\left[\begin{array}{l}a\,(m - n + 1)\,x^{m-n} + \\ b\,(m + nr - ns + 1)\,x^m\end{array}\right]}{bns}\right\}(a + bx^n)^{r-s}\,dx.$$

Now let the value of s be such that

$$m + nr - ns + 1 = 0 \text{ or } s = \frac{m + nr + 1}{n};$$

then $$dv = \frac{a\,(m - n + 1)\,(a + bx^n)^{(-m-1)/n} x^{m-n} dx}{b\,(m + nr + 1)}.$$

Substituting the values of v, z, dv, and dz in (1) and integrating,

$$\int (a + bx^n)^r x^m dx =$$

$$\frac{(a + bx^n)^{r+1} x^{m-n+1} - a\,(m - n + 1) \int (a + bx^n)^r x^{m-n} dx}{b\,(m + nr + 1)}, \tag{A}$$

in which the integral of (2) is made to depend upon that of

$$(a+bx^n)^r x^{m-n}dx.$$

In a similar manner it will be found that

$$\int (a+bx^n)^r x^{m-n}dx$$

depends upon $\int (a+bx^n)^r x^{m-2n}dx;$

and by continuing the process, the exponent of x without the parenthesis can be diminished until it is less than n.

Hence *the integral of a binomial rate may be made to depend upon the integral of another rate of the same form, but in which the exponent of the variable without the parenthesis is diminished by the exponent of the variable within.*

If the rate is of the form

$$du=(a^2-x^2)^{-\frac{1}{2}} x^m dx,$$

substituting a^2 for a, -1 for b, 2 for n, and $-\frac{1}{2}$ for r in formula (A) gives

$$u=\int (a^2-x^2)^{\frac{1}{2}} x^m dx = -\frac{1}{m}(a^2-x^2)^{\frac{1}{2}} x^{m-1} +$$

$$\frac{a^2(m-1)}{m}\int (a^2-x^2)^{-\frac{1}{2}} x^{m-2}dx. \qquad (a)$$

In this $\int (a^2-x^2)^{-\frac{1}{2}} x^m dx$

depends on $\int (a^2-x^2)^{-\frac{1}{2}} x^{m-2}dx,$

and this, by a similar process, will be found to depend upon

$$\int (a^2-x^2)^{-\frac{1}{2}} x^{m-4}dx,$$

and so on; so that after $\frac{1}{2}m$ operations, since m is an even number, the integral will depend upon

$$\int (a^2-x^2)^{-\frac{1}{2}} dx, \qquad (4)$$

which is, by Art. 76, $\sin^{-1}\frac{x}{a}$.

If $du=(a^2+x^2)^{-\frac{1}{2}} x^m dx,$

by substituting in formula (A), a^2 for a, 1 for b, 2 for n, and $-\frac{1}{2}$ for r, the result will be

$$u = \int (a^2 + x^2)^{-\frac{1}{2}} x^m dx = \frac{1}{m} (a^2 + x^2)^{\frac{1}{2}} x^{m-1}$$

$$- \frac{a^2 (m-1)}{m} \int (a^2 + x^2)^{-\frac{1}{2}} x^{m-2} dx, \qquad (b)$$

in which $\int (a^2 + x^2)^{-\frac{1}{2}} x^m dx$

depends on $\int (a^2 + x^2)^{-\frac{1}{2}} x^{m-2} dx,$

and by continuing the process, when m is even, the integral will depend on

$$\int (a^2 + x^2)^{-\frac{1}{2}} dx. \qquad (5)$$

Assuming $v = x + (a^2 + x^2)^{\frac{1}{2}},$

then $dv = dx + (a^2 + x^2)^{-\frac{1}{2}} x dx = \dfrac{x + (a^2 + x^2)^{\frac{1}{2}}}{(a^2 + x^2)^{\frac{1}{2}}} dx,$

and $\dfrac{dv}{v} = \dfrac{dx}{(a^2 + x^2)^{\frac{1}{2}}} = (a^2 + x^2)^{-\frac{1}{2}} dx;$

therefore $\int (a^2 + x^2)^{-\frac{1}{2}} dx = \log \{x + (a^2 + x^2)^{\frac{1}{2}}\} + C.$

If $du = (2ax - x^2)^{-\frac{1}{2}} x^m dx, \qquad (6)$

assume $v = (2ax - x^2)^{\frac{1}{2}} x^{m-1} = (2ax^{2m-1} - x^{2m})^{\frac{1}{2}};$

then $dv = \dfrac{a(2m-1) x^{2m-2} dx - m x^{2m-1} dx}{(2ax^{2m-1} - x^{2m})^{\frac{1}{2}}} =$

$$\frac{a(2m-1) x^{m-1} dx}{(2ax - x^2)^{\frac{1}{2}}} - \frac{m x^m dx}{(2ax - x^2)^{\frac{1}{2}}}.$$

But the last term is equal to mdu; therefore

$$dv = \frac{a(2m-1) x^{m-1} dx}{(2ax - x^2)^{\frac{1}{2}}} - mdu$$

or $du = -\dfrac{dv}{m} + \dfrac{a(2m-1) x^{m-1} dx}{m (2ax - x^2)^{\frac{1}{2}}}.$

Hence, by integrating and substituting for v its value, it will be found that

$$\int \frac{x^m dx}{(2ax - x^2)^{\frac{1}{2}}} = -\frac{1}{m} (2ax - x^2)^{\frac{1}{2}} x^{m-1} +$$

$$\frac{a(2m-1)}{m} \int \frac{x^{m-1} dx}{(2ax - x^2)^{\frac{1}{2}}}, \qquad (c)$$

in which $\int \frac{x^m dx}{(2ax - x^2)^{1/2}}$ depends on $\int \frac{x^{m-1} dx}{(2ax - x^2)^{1/2}}$, and this can be found to depend on

$$\int \frac{x^{m-2} dx}{(2ax - x^2)^{1/2}},$$

and so on; so that after m operations, when m is a positive whole number, the integral will depend on

$$\int \frac{dx}{(2ax - x^2)^{1/2}},$$

which is, by Art. 76,

$$\text{vers}^{-1} \frac{x}{a}.$$

In order to obtain formulas when m is negative, multiply formula (A) by $b\,(m + nr + 1)$; then

$$b\,(m + nr + 1) \int (a + bx^n)^r\, x^m dx =$$

$$(a + bx^n)^{r+1}\, x^{m-n+1} - a\,(m - n + 1) \int (a + bx^n)^r\, x^{m-n} dx.$$

Transposing the terms containing the sign of integration and dividing by $a\,(m - n + 1)$ give

$$\int (a + bx^n)^r\, x^{m-n}\, dx =$$

$$\frac{(a + bx^n)^{r+1}\, x^{m-n+1} - b\,(m + nr + 1) \int (a + bx^n)^r\, x^m dx}{a\,(m - n + 1)} \qquad (B)$$

in which $\int (a + bx^n)^r\, x^{m-n} dx$

depends on $\int (a + bx^n)^r\, x^m dx.$

In a similar manner it will be found that

$$\int (a + bx^n)^r\, x^m dx$$

depends on $\int (a + bx^n)^r\, x^{m+n} dx.$

and, finally, the exponent of x without the parenthesis of the last term of (B) can be increased until it is less or greater than n and positive.

Substituting a^2 for a, ± 1 for b, and 2 for n in (B), it will be seen that

$$\int (a^2 + x^2)^r x^{m-2} dx =$$
$$\frac{(a^2 \pm x^2)^{r+1} x^{m-1} \pm (m + 2r + 1) \int (a^2 \pm x^2)^r x^m dx}{a^2 (m - 1)}. \quad (d)$$

80. Again resuming

$$du = (a + bx^n)^r x^m dx,$$

and assuming $z = x^s$,

in which such a value may be assigned to the exponent s as may be desired; then, passing to the rate,

$$dz = sx^{s-1} dx. \quad (1)$$

Assume $vdz = (a + bx^n)^r x^m dx$;

then, dividing it by (1),

$$v = \frac{1}{s} (a + bx^n)^r x^{m+s+1},$$

the rate of which is

$$dv = \frac{1}{s} (m - s + 1) (a + bx^n)^r x^{m-s} dx +$$
$$\frac{bnr}{s} (a + bx^n)^{r-1} x^{m-s+n} dx.$$

But $(a + bx^n)^r = (a + bx^n) (a + bx^n)^{r-1}$;

hence $dv = \frac{1}{s} \{a (m - s + 1) + b (m - s + nr + 1) x^n\}$

$$(a + bx^n)^{r-1} x^{m-s} dx.$$

Now let the value of s be such that

$$m - s + nr + 1 = 0 \text{ or } s = m + nr + 1;$$

then $dv = \dfrac{-anr (a + bx^n)^{r-1} x^{m-s} dx}{m + nr + 1}$.

Substituting the values v, z, dv, and dz in (1) of the last article, the result is

$$u = \int (a + bx^n)^r x^m dx =$$
$$\frac{(a + bx^n)^r x^{m+1} + anr \int (a + bx^n)^{r-1} x^m dx}{m + nr + 1}, \quad (C)$$

in which $\int (a + bx^n)^r x^m dx$

depends on $\int (a + bx^n)^{r-1} x^m dx,$

and this, by a like process, will be found to depend on

$$\int (a + bx^n)^{r-2} x^m dx,$$

and so on, till r, the exponent of the parenthesis of the term containing the sign of integration, will be reduced to less than unity when positive.

To obtain a formula when r is negative, multiply (C) by $m + nr + 1$, transpose the terms containing the sign of integration, and divide by anr; then

$$u = \int (a + bx^n)^{r-1} x^m dx =$$

$$\frac{-(a + bx^n)^r x^{m+1} + (m + nr + 1) \int (a + bx^n)^r x^m dx}{anr}. \quad (D)$$

In (D) $\int (a + bx^n)^{r-1} x^m dx$

depends on $\int (a + bx^n)^r x^m dx,$

and, by repeating the process, can be made to depend upon a rate in which the exponent of $(a + bx^n)$ will be positive.

EXAMPLES

Determine the integrals of the following:

1. $du = (a - x^2)^3 x^3 dx$
2. $du = \dfrac{x^2 dx}{(1 - x^2)^{\frac{1}{2}}}$
3. $du = -(a + bx^2)^{-\frac{1}{2}} x^2 dx$
4. $du = (1 + x)^{-\frac{1}{2}} x^{-2} dx$

Rational Fractional Rates

81. Every rational fractional rate can be reduced to the form

$$(px^m + qx^{m-1} \ldots . + rx + s)\, dx + \frac{A'x^{n-1}dx + B'x^{n-2}dx \ldots . + R'xdx + S'dx}{Ax^n + Bx^{n-1} \ldots . + Rx + s},$$

in which the exponents of the variable are all positive whole numbers, and the greatest in the numerator of the fraction is at least one less than in the denominator. Hence, since that part of the expression which is not fractional can readily be integrated, it only remains to integrate the fractional part, or

$$du = \frac{A'x^{n-1}dx + B'x^{n-2}dx \ldots + R'xdx + S'dx}{Ax^n + Bx^{n-1} \ldots + Rx + S}. \quad (1)$$

By resolving the denominator of this fraction into factors of the first degree, and assuming them to be

$$x - a, \; x - b, \; x - c, \text{ etc.},$$

the equation may be written under the form

$$du = \frac{Edx}{x-a} + \frac{Fdx}{x-b} + \frac{Gdx}{x-c} \ldots + \frac{Kdx}{x-k}, \quad (2)$$

in which E, F, G, etc. are arbitrary constants whose values can be determined in terms of a, b, c, etc. and A', B', C', etc. by reducing (2) to a common denominator, and comparing the coefficients of the like powers of x in the numerator of the resulting fraction with those in the numerator of (1).

Hence, when no two or more factors of the denominator of (1) are alike, the integral of (2) is, by Art. 72,

$$u = E \log(x - a) + F \log(x - b) +$$
$$G \log(x - c) \ldots + K \log(x - k) + C. \quad (3)$$

When, however, two or more factors are equal, as $a = b = c$, (2) becomes

$$du = \frac{Edx}{x-a} + \frac{Fdx}{x-a} + \frac{Gdx}{x-a} \ldots + \frac{Kdx}{x-k}, \quad (4)$$

in which E, F, and G have the same denominator; consequently these can be represented by a single constant, as in

$$du = \frac{Hdx}{(x - a)} \ldots + \frac{Kdx}{x - k}.$$

Here it will be seen that there are two more equations to satisfy than there are arbitrary constants to be determined; this condition, however, can be obviated by writing the equation thus:

$$du = \frac{Edx}{(x - a)^3} + \frac{Fdx}{(x - a)^2} + \frac{Gdx}{x - a} \ldots + \frac{Kdx}{x - k}, \quad (5)$$

which retains the common denominator of (4).

In like manner, if there are two or more factors, as $(x - a)^m$, $(x - b)^r$, the equation can be written thus:

$$du = \frac{Edx}{(x-a)^m} + \frac{Fdx}{(x-a)^{m-1}} \cdots + \frac{Gdx}{(x-b)^r} + \frac{Hdx}{(x-b)^{r-1}} \cdots + \frac{Kdx}{x-k}.$$

The terms $\frac{Edx}{(x-a)^3}$, $\frac{Fdx}{(x-a)^2}$ in (5), also $\frac{Edx}{(x-b)^m}$, $\frac{Fdx}{(x-b)^{m-1}}$, etc. in this equation are equivalent to $E(x-a)^{-3}dx$, $F(x-a)^{-2}dx$; and $E(x-b)^{-m}dx$, $F(x-b)^{-m+1}dx$, etc. can be integrated by Art. 74; and the terms having denominators of the first power, by logarithms.

If $$du = \frac{ax^2dx - c^3dx}{x^3 - c^2x}, \quad (6)$$

the factors of the denominator are

$$x,\ x-e, \text{ and } x+c;$$

therefore $$du = \frac{ax^2dx - c^3dx}{x(x-c)(x+c)}.$$

Making $$du = \frac{Edx}{x} + \frac{Fdx}{x-c} + \frac{Gdx}{x+c}, \quad (7)$$

and reducing it to a common denominator give

$$du = \frac{Ex^2dx - Ec^2dx + Fx^2dx + Fcxdx + Gx^2dx - Gcxdx}{x^3 - c^2x}.$$

Comparing the numerator of this with that of (6), it will be found that

$$E + F + G = a,\ Fc - Gc = 0, \text{ and } Ec^2 = c^3,$$

whence $E = c$, $F = \frac{1}{2}(a-c)$, and $G = \frac{1}{2}(a-c)$.

Substituting these values in (7) gives

$$du = \frac{cdx}{x} + \frac{\frac{1}{2}(a-c)\,dx}{x-c} + \frac{\frac{1}{2}(a-c)\,dx}{x+c};$$

and integrating,

$$u = c \log x + \frac{1}{2}(a - c)\log(x - c) + \frac{1}{2}(a - c)\log(x + c) + C$$

$$= c \log x + \frac{1}{2}(a - c)\log(x - c)(x + c) + C$$

$$= c \log x + \frac{1}{2}(a - c)\log(x^2 - c^2) + C$$

$$= c \log x + (a - c)\log(x^2 - c^2)^{\frac{1}{2}} + C.$$

If $$du = \frac{2xdx - 7dx}{(x-1)^2(x-2)}, \qquad (8)$$

then [see (4) and (5)]

$$du = \frac{Edx}{(x-1)^2} + \frac{Fdx}{x-1} + \frac{Gdx}{x-2}. \qquad (9)$$

Reducing to a common denominator,

$$du = \frac{E(x-2) + F(x^2 - 3x + 2) + G(x^2 - 2x + 1)}{(x-1)^2(x-2)}dx.$$

Comparing the numerator of this with that of (8), the following are obtained:

$$E = 5,\ F = 3, \text{ and } G = -3.$$

Substituting these values in (9) gives

$$du = \frac{5dx}{(x-1)^2} + \frac{3dx}{x-1} - \frac{3dx}{x-2}$$

$$= 5(x-1)^{-2}dx + \frac{3dx}{x-1} - \frac{3dx}{x-2},$$

and integrating by Arts. 74 and 72,

$$u = -5(x-1)^{-1} + 3\log(x-1) - 3\log(x-2) + C.$$

To verify the principle set forth in (5), let

$$du = \frac{(ax^2 + bx + c)\,dx}{(x - r)^3}.$$

Assume $x - r = v$; then $x = v + r$, $dx = dv$, and

$$du = \frac{a\,(v^2 + 2rv + r^2)\,dv + b\,(v + r)\,dv + cdv}{v^3},$$

or, collecting like powers of v,

$$du = \frac{(ar^2 + br + c)\,dv + (2ar + b)\,vdv + av^2dv}{v^3},$$

and, reducing,

$$du = \frac{(ar^2 + br + c)\,dv}{v^3} + \frac{(2ar + b)\,dv}{v^2} + \frac{adv}{v}.$$

Substituting for v and dv their values, $x - r$ and dx, then

$$du = \frac{(ar^2 + br + c)\,dx}{(x - r)^3} + \frac{(2ar + b)\,dx}{(x - r)^2} + \frac{adx}{x - r}, \quad (10)$$

in which $ar^2 + br + c$ is represented in (5) by E, $2ar + b$ by F, and a by G.

Integrating (10) by Arts. 74 and 72 gives

$$u = -\frac{1}{2}\,(ar^2 + br + c)\,(x - r)^{-2} -$$

$$(2ar + b)\,(x - r)^{-1} + a \log\,(x - r).$$

82. When the denominator contains a single pair of imaginary factors, as $x + r + s\sqrt{-1}$ and $x + r - s\sqrt{-1}$ (whose product is $x^2 + 2rx + r^2 + s^2$), the fraction becomes

$$du = \frac{A'x^{n-1}dx + B'x^{n-2}dx \ldots + S'dx}{(Ax^{n-2} + Bx^{n-3} \ldots + S)\,(x^2 + 2rx + r^2 + s^2)}, \quad (1)$$

which, assuming the factors of the denominator, other than the imaginary pair, to be $x - a$, $x - b$, etc., may be written thus:

$$du = \frac{Edx}{x - a} + \frac{Fdx}{x - b} + \ldots + \frac{Kdx}{x - k} +$$

$$\frac{Pxdx + Qdx}{x^2 + 2rx + r^2 + s^2}. \quad (2)$$

By reducing this to a common denominator and comparing the numerator with that of (1), the values of E, F, etc., also of P and Q, may be determined.

All but the last term of the second member of (2) can be integrated by the methods in Art. 81; therefore it will only be necessary to integrate the last term, which may be put under the form

$$dv = \frac{(Px + Q)\,dx}{(x + r)^2 + s^2}. \tag{3}$$

Assume $x + r = z$; then since $dx = dz$, (3) becomes

$$dv = \frac{(Pz - Pr + Q)\,dz}{z^2 + s^2} = \frac{Pzdz}{z^2 + s^2} - \frac{(Pr - Q)\,dz}{z^2 + s^2},$$

the integral of which is by Arts. 72 and 76

$$v = \frac{1}{2} P \log (z^2 + s^2) - \frac{Pr - Q}{s} \tan^{-1} \frac{z}{s} + C.$$

Therefore, substituting for z its value $x + r$, the integral of (3) is found to be

$$v = \frac{1}{2} P \log (x^2 + 2rx + r^2 + s^2) - \frac{Pr - Q}{s} \tan^{-1} \frac{x + r}{s} + C$$

or

$$v = P \log (x^2 + 2rx + r^2 + s^2)^{\frac{1}{2}} - \frac{Pr - Q}{s} \tan^{-1} \frac{x + r}{s} + C.$$

If $$du = \frac{(2 - x)\,dx}{x^3 + 1} = \frac{(2 - x)\,dx}{(x + 1)\,(x^2 - x + 1)}; \tag{4}$$

then $$du = \frac{Edx}{x + 1} + \frac{(Px + Q)\,dx}{x^2 - x + 1} = \frac{Edx}{x + 1} +$$

$$\frac{Pxdx}{x^2 - x + 1} + \frac{Qdx}{x^2 - x + 1}, \tag{5}$$

whence it is found that $E = 1$, $P = -1$, and $Q = 1$.

Substituting z for $x - \frac{1}{2}$, (5) becomes

$$du = \frac{dz}{z + \frac{3}{2}} - \frac{(z + \frac{1}{2})\,dz}{z^2 + \frac{3}{4}} + \frac{dz}{z^2 + \frac{3}{4}} =$$

$$\frac{dz}{z+\frac{3}{2}}-\frac{zdz}{z^2+\frac{3}{4}}+\frac{\frac{1}{2}dz}{z^2+\frac{3}{4}},$$

the integral of which is

$$u=\log\left(z+\frac{3}{2}\right)-\frac{1}{2}\log\left(z^2+\frac{3}{4}\right)+\frac{\sqrt{3}}{3}\tan^{-1}\frac{2z\sqrt{3}}{3};$$

$$u=\log(x+1)-\frac{1}{2}\log(x^2-x+1)+$$
$$\frac{\sqrt{3}}{3}\tan^{-1}\frac{(2x-1)\sqrt{3}}{3}$$

or

$$u=\log\frac{x+1}{(x^2-x+1)^{\frac{1}{2}}}+\frac{\sqrt{3}}{3}\tan^{-1}\frac{(2x-1)\sqrt{3}}{3}+C.$$

When the denominator contains several sets of imaginary factors, respectively equal to each other, the factor $x+2rx+r+s$ will enter the denominator several times; hence, for that part of the fraction containing only sets of equal imaginary factors, may be put under the following form, thus

$$du=\frac{(Ex+F)\,dx}{(x^2+2rx+r^2+s^2)^m}+\frac{(Gx+H)\,dx}{(x^2+2rx+r^2+s^2)^{m-1}}\cdots+$$
$$\frac{(Px+Q)\,dx}{(x^2+2rx+r^2+s^2)}. \quad (6)$$

The values of the constants E, F, G, etc., may be determined as heretofore explained; then the integral of each term taken separately.

Since the terms of the second member of (6) are all of the same general form, it will only be necessary to integrate the first term, which may be placed under the form

$$dv=\frac{Exdx+Fdx}{(x^2+2rx+r^2+s^2)^m}. \quad (7)$$

Assuming $x=z-r$, this expression becomes

$$dv=\frac{Ezdz-(Er-F)\,dz}{(z^2+s^2)^m}=$$

$$Ez\,(z^2+s^2)^{-m}\,dz - (Er-F)\,(z^2+s^2)^{-m}\,dz,$$

and, by Art. 74,

$$\int Ez\,(z^2+s^2)^{-m}\,dz = \frac{E\,(z^2+s^2)^{1-m}}{2\,(1-m)}.$$

By formula (D) of Art. 80, $\int -(Er-F)\,(z^2+s^2)^{-m}dz$ can be made to depend upon $\int -(Er-F)\,(z^2+s^2)^{-1}\,dz$, which is $-\dfrac{(Er-F)}{s}\tan^{-1}\dfrac{z}{s}$, thus completing the integration of (7).

From the preceding, it is evident that the fraction can be integrated, even when the denominator contains several different imaginary factors; providing, however, said factors can be determined, and this condition applies to all fractional rates.

EXAMPLES

1. $du = \dfrac{adx}{x^2 - a^2}$

2. $du = \dfrac{2axdx + adx}{x^3 - 1}$

3. $du = \dfrac{2\,(1-x)\,dx}{(1+x^2)^2}$

Irrational Fractional Rates

83. Any irrational fractional rate will admit of integration when it can be changed to a rational form. Thus, let

$$du = \frac{(x^{2/3} + ax^{1/6} + b)\,dx}{x + cx^{1/2} + e},$$

and assume $x = z^6$; then

$$du = \frac{(z^4 + az + b)\,6z^5dz}{z^6 + cz^3 + e} = \frac{(6z^9 + 6az^6 + 6bz^5)dz}{z^6 + cz^3 + e},$$

which is a rational form and consequently can be integrated by the methods explained in Arts. 81 and 82.

When the quantity under the radical sign is a polynomial, the rate can not in general be changed to one of a rational form. If, however, the rate is of the form

$$du = (a^2 + bx + c^2x^2)^{1/2}\,Xdx, \qquad (1)$$

in which X is a rational function of x, it can be changed to a rate which will be rational; thus, assuming

$$(a^2 + bx + c^2x^2)^{\frac{1}{2}} = z - cx, \tag{2}$$

then $$a^2 + bx + c^2x^2 = z^2 - 2czx + c^2x^2;$$

whence $$x = \frac{z^2 - a^2}{2cz + b}. \tag{3}$$

This value of x substituted in the second member of (2), by reducing, gives

$$(a^2 + bx + c^2x^2)^{\frac{1}{2}} = \frac{cz^2 + bz + a^2c}{2cz + b}. \tag{4}$$

The rate of (3) is

$$dx = \frac{2\,(cz^2 + bz + a^2c)\,dz}{(2cz + b)^2}. \tag{5}$$

(5) divided by (4) gives

$$(a^2 + bx + c^2x^2)^{-\frac{1}{2}}\,dx = \frac{2dz}{2cz + b}; \tag{6}$$

hence $$du = (a^2 + bx + c^2x^2)^{-\frac{1}{2}}\,Xdx = \frac{2Xdz}{2cz + b}, \tag{7}$$

which is a rational form; for, since X is a rational function of x, it must also be a rational function of z; that is, if the value of x be substituted in X, it will give the value of X in rational terms of z.

If $X = \frac{1}{x}$, then substituting this value of X in (7), since $x = \frac{z^2 - a^2}{2cz + b}$ [see (3)], (7) becomes

$$du = \left(\frac{2dz}{2cz + b}\right)\left(\frac{2cz + b}{z^2 - a^2}\right) = \frac{2dz}{z^2 - a^2}.$$

The integral of this is, by Art. 80,

$$u = \frac{1}{a}\log\frac{z - a}{z + a};$$

but from (2), $z = (a^2 + bx + c^2a^2)^{\frac{1}{2}} + cx$,

therefore $$u = \frac{1}{a}\log\frac{(a^2 + bx + c^2x^2)^{\frac{1}{2}} + cx - a}{(a^2 + bx + c^2x^2)^{\frac{1}{2}} + cx + a} + C.$$

If $X = 1$, the integral of (7) will be

$$u = \frac{1}{c} \log (2cz + b);$$

but [see (2)],

$$2cz + b = 2c\,(a^2 + bx + c^2x^2)^{\frac{1}{2}} + 2c^2x + b;$$

therefore

$$u = \frac{1}{c} \log \{2c\,(a^2 + bx + c^2x^2)^{\frac{1}{2}} + 2c^2x + b\} + C.$$

If $b = 0$, then

$$u = \frac{1}{c} \log 2c\,\{(a^2 + c^2x^2)^{\frac{1}{2}} + cx\} + C.$$

If $X = x$ then (7) becomes

$$du = \frac{2\,(z^2 - a^2)\,dz}{(2cz + b)^2},$$

which can be integrated by Art. 81.

If $$du = \frac{(a^2 + bx + x^2)^{\frac{1}{2}}\,dx}{X},$$

assume $$(a^2 + bx + x^2)^{\frac{1}{2}} = x + z;$$

then [see (3) and (4)]

$$x = \frac{z^2 - a^2}{b - 2z} \qquad (8)$$

and $$(a^2 + bx + x^2)^{\frac{1}{2}} = -\frac{z^2 - bz + a^2}{b - 2z}. \qquad (9)$$

Taking the rate of (8) and reducing [see (5)]

$$du = -\frac{2\,(z^2 - bz + a^2)\,dz}{(b - 2z)^2}. \qquad (10)$$

Multiplying (10) by (9) gives

$$(a^2 + bx + x^2)^{\frac{1}{2}}\,dx = \frac{2\,(z^2 - bz + a^2)^2\,dz}{(b - 2z)^3};$$

therefore $$du = \frac{2\,(z^2 - bz + a^2)^2\,dz}{(b - 2z)^3\,X},$$

which is rational in terms of z, as previously explained.

84. When the rate is of the form

$$du = \frac{Xdx}{(c + dx - x^2)^{1/2}},$$

assume $c = ab$ and $d = a - b$; then

$$du = \frac{Xdx}{\{ab + (a - b)\,x - x^2\}^{1/2}}.$$

Now, since $ab + (a - b)\,x - x^2 = (a - x)\,(b + x)$, assume

$$\sqrt{[(a - x)\,(b + x)]} = (a - x)\,z; \qquad (1)$$

then, squaring both members,

$$(a - x)\,(b + x) = (a - x)^2\,z^2$$

or

$$b + x = (a - x)\,z^2,$$

whence

$$x = \frac{az^2 - b}{z^2 + 1}; \qquad (2)$$

and therefore,

$$a - x = a - \frac{az^2 - b}{z^2 + 1} = \frac{a + b}{z^2 + 1}. \qquad (3)$$

Substituting this value of $a - x$ in the second member of (1), the result is

$$\sqrt{[(a - x)\,(b + x)]} = \frac{(a + b)\,z}{z^2 + 1}. \qquad (4)$$

The rate of (2) is

$$dx = \frac{2\,(a + b)\,zdz}{(z^2 + 1)^2}. \qquad (5)$$

Dividing this by (4) and reducing give

$$\frac{dx}{\sqrt{[(a - x)\,(b + x)]}} = \frac{2dz}{z^2 + 1}.$$

Therefore, multiplying both members by X, it is found that

$$du = \frac{Xdx}{\sqrt{[(a - x)\,(b + x)]}} = \frac{2Xdz}{z^2 + 1},$$

which is rational in terms of z, as shown in Art. 83. When $X = 1$, this becomes

$$du = \frac{2dz}{z^2 + 1};$$

Hence $$u = 2\tan^{-1} z + C.$$

If $$du = \frac{\sqrt{[(a-x)(b+x)]}\,dx}{X}, \qquad (6)$$

then, proceeding as before, it will be found that

$$du = \frac{2(a+b)^2 z^2 dz}{X(z^2+1)^3}, \qquad (7)$$

which is also a rational fraction.

Let $$du = \frac{(x-x^2)^{\frac{1}{2}}\,dx}{(1-x)^2}. \qquad (8)$$

Here $a = 1$, $b = 0$, $(1-x)^2 = \frac{1}{(z^2+1)^2}$, $(x-x^2)^{\frac{1}{2}} = \frac{z}{z^2+1}$, and $dx = \frac{2z dz}{(z^2+1)^2}$ [see (2), (3), (4), and (5)]; therefore, substituting these values in (8) and reducing, it will be found that

$$du = \frac{2z^2 dz}{z^2+1} = 2\,dz - \frac{2dz}{z^2+1},$$

the integral of which is

$$u = 2z - 2\tan^{-1} z + C.$$

Substituting for z its value, $\left(\frac{x}{1-x}\right)^{\frac{1}{2}}$,

$$u = 2\left(\frac{x}{1-x}\right)^{\frac{1}{2}} - 2\tan^{-1}\left(\frac{x}{1-x}\right)^{\frac{1}{2}} + C.$$

EXAMPLES

1. $du = (x^2+a)^{\frac{1}{2}}\,dx$

2. $du = -\frac{dx}{(1-x^2)^{\frac{1}{2}}}$

3. $du = \frac{3(x-x^2)^{\frac{1}{2}}\,dx}{x^3}$

Transcendental Rates

85. Simple rates of this class, which admit of direct integration, have been previously treated; a few of those whose integrals are less readily obtained will now be considered, omitting the constant C.

Let $$du = Xa^x dx,$$

in which X is an algebraic function of x, and its nth ratal coefficient is constant, represented by A in the formula.

Assume $v = X$, $dz = a^x dx$, and $\dfrac{dX}{dx} = X'$, $\dfrac{dX'}{dx} = X''$, etc.,

then $$dv = dX \text{ and } z = \frac{a^x}{\log a}.$$

These values of v, z, dv, and dz in (1), Art. 79, give

$$\int Xa^x dx = \frac{Xa^x}{\log a} - \frac{1}{\log a}\int dXa^x,$$

$$-\frac{1}{\log a}\int dXa^x = -\frac{X'a^x}{(\log a)^2} + \frac{1}{(\log a)^2}\int dX'a^x,$$

and $$\frac{1}{(\log a)^2}\int dX'a^x = \frac{X''a^x}{(\log a)^3} - \frac{1}{(\log a)^3}\int dX''a^x, \text{ etc.};$$

from which the following is obtained:

$$u = a^x \left\{\frac{X}{\log a} - \frac{X'}{(\log a)^2} + \frac{X''}{(\log a)^3} \cdots\cdots \pm \frac{A}{(\log a)^n}\right\}. \quad (1)$$

EXAMPLE

$$du = (bx^2 + cx^4)a^x.$$

Here $X = bx^2 + cx^4$, the ratal coefficients of which are

$$2bx + 4cx^3,\ 2x + 12cx^2,\ 24cx, \text{ and } 24c = A.$$

Substituting these values in (1) gives

$$u = a^x \left\{\frac{bx^2 + cx^4}{\log a} - \frac{2bx + 4cx^3}{(\log a)^2} + \right.$$

$$\left. \frac{2b + 12cx^2}{(\log a)^3} - \frac{24cx}{(\log a)^4} + \frac{24c}{(\log a)^5}\right\},$$

from which it will be seen that when the greatest exponent of x is even, the sign of the last term of (1) will be positive, and when it is odd, the sign of the last term will be negative.

If $$du = x^m a^x dx,$$

then $X = x^m$, whose ratal coefficients are (m being a positive number):

$$mx^{m-1},\ m(m-1)\,x^{m-2},\ m(m-1)(m-2)\,x^{m-3}, \text{ etc.}$$

Substituting these values in (1) gives

$$u = a^x \{\frac{x^m}{\log a} - \frac{mx^{m-1}}{(\log a)^2} + \frac{m(m-1)x^{m-2}}{(\log a)^3} - \frac{m(m-1)(m-2)x^{m-3}}{(\log a)^4} + \dots \pm \frac{m(m-1)\dots 1}{(\log a)^{m+1}}\}. \quad (2)$$

If m be negative or fractional, then develop a^x by Maclaurin's theorem, Art. 24, multiply both members by x^m and integrate.

86. When the rate is in the form of a logarithm, as

$$du = x^n \log x dx,$$

assume $v = \log x$ and $dz = x^n dx$;

then $dv = \frac{dx}{x}$ and $z = \frac{x^{n+1}}{n+1}$.

These values of v, z, dv, and dz in (1), Art. 79, give

$$du = \int x^n \log x dx = \frac{x^{n+1}}{n+1} \log x - \int (\frac{x^{n+1}}{n+1}) \frac{dx}{x};$$

but $\int (\frac{x^{n+1}}{n+1}) \frac{dx}{x} = \int \frac{x^n dx}{n+1} = \frac{x^{n+1}}{(n+1)^2}$;

therefore $u = \frac{x^{n+1}}{n+1} (\log x - \frac{1}{n+1})$.

Let $du = (\log x)^n dx, \quad (1)$

in which n is a positive integer.

Assume $v = (\log x)^n$ and $dz = dx$;

then $dv = n (\log x)^{n-1} \frac{dx}{x}$ and $z = x$;

and, by substitution,

$$du = \int (\log x)^n dx = x (\log x)^n - n \int (\log x)^{n-1} dx;$$

but

$$-n \int (\log x)^{n-1} dx = -nx (\log x)^{n-1} + n(n-1) \int (\log x)^{n-2} dx,$$

and $n(n-1) \int (\log x)^{n-2} dx =$

$n(n-1)x(\log x)^{n-2} - n(n-1)(n-2)\int(\log x)^{n-3}dx$;

whence $u = x\{(\log x)^n - n(\log x)^{n-1} +$

$$n(n-1)(\log x)^{n-2} \ldots . + n(n-1)(\ldots . 1)\}, \quad (2)$$

in which the last sign will be plus when n is even, and minus when n is odd. If n be negative; that is, if $du = (\log x)^{-n}dx$, assume $dv = dx$ and $z = (\log x)^{-n}$, and proceed as before; u, however, will be found to depend on the integral of $\frac{dx}{\log x}$, sometimes called Soldner's integral, which can be obtained by series.

If $$du = x^m(\log x)^n dx,$$

assume $$y = x^{m+1};$$

then $$\log y = (m+1)\log x$$

or $$(\log x)^n = \left(\frac{1}{m+1}\right)^n(\log y)^n.$$

Therefore, since $dy = (m+1)x^m dx$,

or $$x^m dx = \frac{1}{m+1}dy,$$

$$du = x^m(\log x)^n dx = \left(\frac{1}{m+1}\right)^{n+1}(\log y)^n dy,$$

the integral of which is the same as that of (1) multiplied by $\left(\frac{1}{m+1}\right)^{n+1}$.

If $$du = (\log x)^n X dx,$$

assume $\int X dx = X_1$, $\int \frac{X_1 dx}{x} = X_2$, $\int \frac{X_2 dx}{x} = X_3$, etc.

and $$v = (\log x)^n \text{ and } dz = X dx;$$

then $$dv = n(\log x)^{n-1}\frac{dx}{x} \text{ and } z = \int X dx = X_1.$$

These values of v, z, dv, and dz substituted in (1), Art. 79, give $u = (\log x)^n X_1 - n(\log x)^{n-1}X_2 +$

$$n(n-1)(\log x)^{n-2}X_3 - \ldots . n(n-1)(\ldots . 1)X_{n+1}. \quad (3)$$

If the integrals of

$$Xdx, \frac{X_1dx}{x}, \frac{X_2dx}{x}, \ldots \frac{X_ndx}{x},$$

can be found in finite terms, the proposed rate will have an exact integral.

Let $du = (\log x)^3 (1 + x^2)\, dx.$

Here $n = 3, \int Xdx = x + \frac{1}{3} x^3 = X_1,$

$$\int \frac{X_1dx}{x} = x + \frac{1}{9} x^3 = X_2,$$

$$\int \frac{X_2dx}{x} = x + \frac{1}{27} x^3 = X_3 \text{ and } \int \frac{X_3dx}{x} = x + \frac{1}{81} x^3 = X_4.$$

Substituting these values in (3), the result is

$$u = (\log x)^3 (x + \frac{1}{3} x^3) - 3 (\log x)^2 (x + \frac{1}{9} x^3) +$$

$$6 (\log x) (x + \frac{1}{27} x^3) - 6 (x + \frac{1}{81} x^3).$$

Complex Circular Rates

87\. Let $du = \sin^m x \cos^n x dx,$ (1)

and assume $\sin x = v;$

then $\cos x = (1 - v^2)^{\frac{1}{2}}$ and $\cos x dx = dv,$

whence $dx = \frac{dv}{\cos x} = \frac{dv}{(1 - v^2)^{\frac{1}{2}}}.$

Substituting these values of v and dv in (1), gives

$$du = v^m (1 - v^2)^{(n-1)/2} dv. \quad (2)$$

Assuming $\cos x = z,$

then will $du = - z^n (1 - z^2)^{(m-1)/2} dz.$ (3)

These can be integrated by Art. 78: (2), when n is a positive odd integer and m positive or negative, integral or fractional; (3), when m is a positive odd integer, and n positive or negative, integral or fractional. When these conditions do not exist, they can be integrated in many cases by one of the formulas A, B, C, or D.

In $$du = \sin^2 x \cos^3 x\, dx,$$
$m = 2$ and $n = 3$; therefore (2) becomes
$$du = v^2 (1 - v^2)\, dv,$$
hence $$u = \frac{1}{3} v^3 - \frac{1}{5} v^5 = (5 - 3v^2) \frac{v^3}{15}.$$
or, substituting the value of v,
$$u = (5 - 3 \sin^2 x) \frac{\sin^3 x}{15}.$$

(3) also becomes
$$du = -(1 - z^2)^{\frac{1}{2}} z^3 dz,$$
which by formula A can be made to depend on
$$\int (1 - z^2)^{\frac{1}{2}} z dz,$$
which is integrable by Art. 74.

In $$du = \sin x^n\, dx,$$
assume $\sin x = v$; then, since $\cos x = (1 - v^2)^{\frac{1}{2}}$ and $dx = \frac{dv}{\cos x}$,
$$du = (1 - v^2)^{-\frac{1}{2}} v^n dv,$$
which, when n is a whole number, either positive or negative, by the application of formula A or B may be made to depend on $(1 - v^2)^{-\frac{1}{2}} dv$ or $(1 - v^2)^{-\frac{1}{2}} vdv.$

The first of these can be integrated by Art. 75 and the second by Art. 74.

If $$du = \tan^n x dx,$$
let $\tan x = v$; then
$$dx = \frac{dv}{1 + v^2},$$
and $$du = \frac{v^n dv}{1 + v^2},$$
a rational fraction.

If $$du = \frac{\tan x dx}{\sin^2 x},$$
then, since $\tan x = \frac{\sin x}{\cos x}$, by substitution the result is

$$du = \frac{dx}{\sin x \cos x};$$

therefore $$u = \log \tan x;$$

(see Art. 32).

Let $$du = \tan^{-1} x dx.$$

Now $$d(x \tan^{-1} x) = \tan^{-1} x dx + \frac{xdx}{1+x^2},$$

and $$\int \frac{xdx}{1+x^2} = \frac{1}{2} \log(1+x^2);$$

therefore $$u = x \tan^{-1} x - \frac{1}{2} \log(1+x^2).$$

Let $$du = X \sin^{-1} x dx,$$
and assume

$$Xdx = dz \text{ and } \sin^{-1} x = v, \text{ also } \int Xdx = X_1;$$

then $$dv = (1-x^2)^{-\frac{1}{2}} dx \text{ and } z = \int Xdx = X_1.$$

Therefore $u = X_1 \sin^{-1} x - \int X_1 (1-x^2)^{-\frac{1}{2}} dx$, in which the integral of the proposed rate is made to depend upon that of another, whose form is algebraic.

A similar process will apply to any of the following forms:

$$X \cos^{-1} x, \ X \tan^{-1} x, \ X \cot^{-1} x, \text{ etc.},$$

since the rates of $\cos^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, etc., all depend upon the integral of an algebraic expression.

Examples

1. $du = a^2x^2dx$
2. $du = \dfrac{x^2dx}{\log^2 x}$
3. $du = \sin^2 x \cos^3 x dx$
4. $du = x \cos^{-1} x dx$

Bernouilli's Series

88. Bernouilli's series expresses the integral of any rate of the form

$$du = Xdx,$$

in which X is a function of x, in terms of X, its ratal coefficients, and x.

To obtain this series, assume

$$X = v \text{ and } dx = dz;$$

then $$dv = dX \text{ and } z = x.$$

Substituting these values in (2) of Art. 79, the result is

$$\int X dx = xX - \int x dX,$$

or, since dx is included in dX,

$$\int X dx = xX - \int x dx \left(\frac{dX}{dx}\right).$$

And assuming $\frac{dX}{dx} = v$ and $x dx = dz$;

then, since $$dv = \frac{d^2X}{dx^2} \text{ and } z = \frac{x^2}{1 \cdot 2},$$

by substituting these values as before,

$$-\int x dx \left(\frac{dX}{dx}\right) = -\frac{x^2}{1 \cdot 2}\left(\frac{dX}{dx}\right) + \int \frac{x^2 dx}{1 \cdot 2}\left(\frac{d^2X}{dx^2}\right).$$

In a similar manner, it will be found that

$$\int \frac{x^2 dx}{1 \cdot 2}\left(\frac{d^2X}{dx^2}\right) = \frac{x^3}{1 \cdot 2 \cdot 3}\left(\frac{d^2X}{dx^2}\right) - \int \frac{x^3 dx}{1 \cdot 2 \cdot 3}\left(\frac{d^3X}{dx^3}\right);$$

therefore, by the substitution of $xX - \frac{x^2}{1 \cdot 2}\left(\frac{dX}{dx}\right) + \frac{x^3}{1 \cdot 2 \cdot 3}\left(\frac{d^2X}{dx^2}\right) -$ etc., the integral of (1) is found to be

$$u = xX - \frac{x^2}{1 \cdot 2}\left(\frac{dX}{dx}\right) + \frac{x^3}{1 \cdot 2 \cdot 3}\left(\frac{d^2X}{dx^2}\right) - \text{etc.}$$

This series was obtained by John Bernouilli in 1694 and is probably the first general development discovered; it is, however, but a particular case of Taylor's theorem, discovered in 1715. Such expressions as $\log(1 + x)$, $\sin x$, and others can be readily developed into a series by Bernouilli's theorem, as shown by him.

Let $$du = (1 + 2x + 3x^2)\, dx,$$

in which $(1 + 2 + 3x^2)$ represents X; then

$$xX = x + 2x^2 + 3x^3,$$

$$-\frac{x^2}{1\cdot 2}\left(\frac{dX}{dx}\right)=-x^2-3x^3, \text{ and } \frac{x^3}{1\cdot 2\cdot 3}\left(\frac{d^2X}{dx^2}\right)=x^3;$$

therefore

$$u=x+2x^2+3x^3-x^2-3x^3+x^3=x+x^2+x^3.$$

EXAMPLES

1. $du=(1+x^2)^{½}\,dx$

2. $du=\dfrac{dx}{1+x^2}$

Successive Integration

89. In the expression

$$d^2u=(x^3+ax^2)\,dx^2,$$

two integrations are required to determine the primitive function, or u in terms of x. Placing the expression under the form

$$\frac{d^2u}{dx}=x^3dx+ax^2dx,$$

and integrating,

$$\frac{du}{dx}=\frac{x^4}{4}+\frac{ax^3}{3}+C.$$

Multiplying through by dx and integrating again,

$$u=\frac{x^5}{4\cdot 5}+\frac{x^4}{3\cdot 4}+C_1x+C_2.$$

The foregoing may be written thus:

$$\frac{d^2u}{dx^2}=f_1(x),$$

$$\frac{du}{dx}=f_2(x)+C_1 \text{ and } u=f_3(x)+C_1x+C_2.$$

From the preceding it will be seen that, if

$$d^nu=f(x)\,dx^n,$$

by taking n successive integration the following will be obtained,

$$u=f_{n+1}(x)+$$

$$\frac{C_1x^{n-1}}{1\cdot 2\ldots\ldots(n-1)}+\frac{C_2x^{n-2}}{1\cdot 2\ldots\ldots(n-2)}+\ldots\ldots C_{n-1}x+C_n.$$

The nth integral of

$$d^n u = f(x)\, dx^n$$

may be represented thus:

$$u = \int f(x) dx^n.$$

Developing $$\frac{d^n u}{dx^n} = f(x)$$

by Maclaurin's theorem (Art. 24), the result is

$$\frac{d^n u}{dx^n} = A + \frac{df(x)\, x}{dx} + \frac{d^2 f(x)\, x^2}{1 \cdot 2 dx^2} + \frac{d^3 f(x)\, x^3}{1 \cdot 2 \cdot 3 dx^3} + \text{etc.}$$

Now, by substituting for A, $\frac{df(x)}{dx}$, $\frac{d^2 f(x)}{dx^2}$, etc., their values as shown in Art. 24, then multiplying by dx and integrating n successive times, plus a constant each integration, the result will be a series expressing the value of u in terms of x.

Let $$\frac{d^4 u}{dx^4} = \frac{1}{1+x},$$

the development of which is

$$\frac{d^4 u}{dx^4} = 1 - x + x^2 - x^3 + \text{etc.}$$

Integrating this as explained, gives

$$u = \frac{x^4}{2 \cdot 3 \cdot 4} - \frac{x^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{x^6}{3 \cdot 4 \cdot 5 \cdot 6} - \text{etc.} + \frac{C_1 x^3}{2 \cdot 3} + \frac{C_2 x^2}{2} + C_3 x + C_4.$$

EXAMPLE

$$d^2 u = 6dx^2 + 36x dx^2 + 30x^4 dx^2.$$

NOTE: In successive integration, it sometimes becomes expedient to integrate between limits, especially when there are two or more independent variables. For instance, in the equation of the circle, $y^2 = 1 - x^2$, x can never be greater than 1 nor less than zero.

Integration of Partial Rates

90. Partial rates are obtained with reference to one variable only, or with reference first to one variable, then to another, etc., (see Art. 22).

Of the first class, as

$$du = 3x^2ydx,$$

the integral is $u = x^3y,$

or, since the primitive function may contain terms in y alone, an arbitrary quantity must be added, as Y, a function of y, as such terms will disappear in passing to the rate; thus,

$$u = x^3y + Y + C.$$

This class of partial rates can be expressed generally thus:

$$d^nu = f\,(x, y, z, \text{etc.})\; dx^n.$$

Taking one of the second class, as

$$d^2u = 9x^2y^2dxdy + 2xdxdy,$$

and integrating first with respect to x, then with respect to y, $du = 3x^3y^2dy + x^2dy + Y$ and $u = x^3y^3 + x^2y + \int Ydy + X + C.$

This class of partial rates can be expressed generally thus:

$$d^mu = f\,(x, y, z, \text{etc.})\; dx^ndy^rdz^s, \text{ etc.},$$

in which m is equal to the sum of the exponents of the rates of the independent variables; that is,

$$m = n + r + s + \text{etc.}$$

Let $$d^2u = 6xy^2dxdy. \qquad (1)$$

The integral of this with respect to x is

$$du = 3x^2y^2dy \qquad (2)$$

or, since there may have been a term containing y alone in (2) which would disappear in (1) by passing to the rate,

$$du = 3x^2y^2dy + Y.$$

Integrating again with respect to y, it will be found that

$$u = x^2y^3 + \int Ydy + X.$$

EXAMPLES

1. $d^2u = a^2x^2dy^2 + ydy^2$
2. $d^3u = x^2ydxdy^2$

Integration of Total Rates

91. Let $du = f_1(x, y)\,dx + f_2(x, y)\,dy,$ (1)
of which the partial rates are

$$du = f_1(x, y)\,dx$$

and

$$du = f_2(x, y)\,dy.$$

Dividing the first by dx and the second by dy give

$$\frac{du}{dx} = f_1(x, y) \quad (2)$$

and

$$\frac{du}{dy} = f_2(x, y) \quad (3)$$

Taking the rate of (2) with respect to y and dividing by dy; then the rate of (3) with respect to x and dividing by dx, the results are

$$\frac{d^2u}{dxdy} = f_3(x, y)$$

and

$$\frac{d^2u}{dydx} = f_4(x, y).$$

Now, as is shown in Art. 23, in order that (1) be integrable,

$$\frac{d^2u}{dxdy} \text{ must equal } \frac{d^2u}{dydx},$$

that is

$$f_3(x, y) = f_4(x, y).$$

This is termed the test of integration.

If $du = f(x, y, z)\,dx + f(x, y, z)\,dy + f(x, y, z)\,dz,$ in order that this expression be integrable the following conditions must be fulfilled: viz.,

$$\frac{d^2u}{dxdy} = \frac{d^2u}{dydx}, \quad \frac{d^2u}{dxdz} = \frac{d^2u}{dzdx}, \quad \frac{d^2u}{dydz} = \frac{d^2u}{dzdy},$$

and similarly if there are four or more independent variables.

Let $du = (3x^2y^2 + y + 1)\,dx + (2x^3 y + x + a)\,dy,$ (4)
the partial rates of which are

$$du = (3x^2y^2 + y + 1)\,dx \quad (5)$$

and

$$du = (2x^3y + x + a)\,dy; \quad (6)$$

whence are obtained the following,

$$\frac{d^2u}{dxdy} = 6x^2y + 1$$

and $$\frac{d^2u}{dydx} = 6x^2y + 1,$$

which fulfill the conditions stated above; therefore (4) is integrable.

It will be seen that the original function must have contained all the terms in x indicated in (5), also all the terms in y indicated in (6). Now the integral of (5) is

$$u = x^3y^2 + xy + x \tag{7}$$

and of (6) $$u = x^3y^2 + xy + ay, \tag{8}$$

but it will be observed that the terms in (8) containing x are also included in (7), and therefore should be omitted in integrating; consequently the integral of (4) is

$$u = x^3y^2 + xy + x + ay.$$

Let $$du = ay^2dx + 2xdy, \tag{9}$$

of which the partial rates are

$$du = ay^2dx \text{ and } du = 2xdy,$$

from which are obtained

$$\frac{d^2u}{dxdy} = 2ay \text{ and } \frac{d^2u}{dydx} = 2,$$

which are not equal; therefore (9) is not integrable.

Let $$du = (2xy + z^2 + 1)dx + (x^2 + 3y^2z + a)\,dy + (2xz + y^3 + 4z^3 + b)\,dz. \tag{10}$$

It is obvious here, as in rates of two independent variables, that the integral of the coefficient of dx must have all the terms containing x in the original function; therefore, in integrating the coefficient of dy, the terms containing x must be omitted, and in integrating the coefficient of dz, the terms containing both x and y must also be omitted.

Proceeding thus, it is found that

$$u = x^2y + xz^2 + x + y^3z + ay + z^4 + bz.$$

EXAMPLES

$$du = (2xy + 3x^2n)\,dx + (2xy + x)\,dy$$

$$du = \frac{ydx}{a - z} + \frac{xdy}{a - z} + \frac{xydz}{(a - z)}$$

$$du = \frac{3}{z}(x^2 - y^2)dx - \frac{2xy}{z}dy + \frac{3xy^2 - x^3}{z^2}dz$$

$$du = (\sin y - y \sin x)\,dx + (\cos x + x \cos y)\,dy$$

Homogeneous Rates

92. A homogeneous rate is one in which the sum of the exponents of the variables is the same in each term; this sum is called the degree of the rate, and is here designated by n.

When such a rate fulfills the conditions given in the last article, the integral can be obtained by substituting, for instance, x, y, z for dx, dy, dz, etc., in their respective factors of the functional rate, thus increasing by unity the exponent each of x, y, z, etc. in its said factor; then collecting like terms and dividing by $n + 1$.

To prove this, let

$$du = Pdx + Qdy + Rdz + \text{etc.} \qquad (1)$$

be a homogeneous rate, in which P, Q, R, etc. are algebraic functions of x, y, z, etc. of the nth degree.

Now it is evident that this must have been deduced from a homogeneous algebraic function of the form

$$u = P'x + Q'y + R'z + \text{etc.}, \qquad (2)$$

of the degree $n + 1$, since taking the rate diminished by unity the exponent of the variable so treated in each term of (1).

Substituting xy' for y, xz' for z, etc. in (2), each term in the value of u will contain x^{n+1}, consequently

$$u = Nx^{n+1}, \qquad (3)$$

in which N is a function of y', z', etc., but does not contain x; hence, the rate of (3) with respect to x, is

$$\frac{du}{dx} = (n + 1)\,Nx^n. \qquad (4)$$

The rates of xy', xz', etc. with respect to x, are $y'dx$, $z'dx$, etc., and these rates substituted in (1) and divided by dx, give

$$\frac{du}{dx} = P + Qy' + Rz' + \text{etc.} \qquad (5)$$

but $\frac{du}{dx} = (n + 1)\,Nx^n$, (4), therefore

$$(n + 1)\,Nx^n = P + Qy' + Rz' + \text{etc.}$$

or, by multiplying by x,

$$(n+1)\,Nx^{n+1} = Px + Qxy' + Rxz' + \text{etc.}$$

Therefore, substituting y for xy', z for xz', etc., and dividing by $(n+1)$,

$$Nx^{n+1} = \frac{Px + Qy + Rz + \text{etc.}}{n+1},$$

or, since $Nx^{n+1} = u$ [see (3)],

$$u = \frac{Px + Qy + Rz + \text{etc.}}{n+1}. \qquad (6)$$

EXAMPLES

Integrate $du = (3x + mxy)\,dx + (x + mxy)\,dy$ and $(nx^{n-1}y + y^n)\,dx + (x^n + xy^{n-1})\,dy + (n+1)\,z^n$.

Length of Curves

93. In case of curves referred to rectangular coördinates, it has been shown in Art. 46 that

$$dz = (dx^2 + dy^2)^{\frac{1}{2}},$$

whence

$$z = \int (dx^2 + dy^2)^{\frac{1}{2}},$$

which is a general expression for the length of a curve, or the length of any arc thereof, estimated from the origin of the coördinates or some special point. When the radical is expressed in terms of x and dx, or y and dy, obtained from the equation of the curve, its integral may be determined.

In case of polar curves, the rate of an arc is [see Art. 57, (1)]:

$$dz = (dr^2 + r^2dv^2)^{\frac{1}{2}},$$

whence

$$z = \int (dr^2 + r^2dv^2)^{\frac{1}{2}},$$

which is the general expression for the length of an arc of a curve referred to polar coördinates, estimated from the pole or some special point. When the radical is expressed in terms of r and dr, or v and dv, its integral may be determined.

Taking the circle whose radius is unity, its sine x and cosine $(1 - x^2)^{\frac{1}{2}}$, then

$$t = \tan z = \frac{x}{(1 - x^2)^{\frac{1}{2}}},$$

whence

$$dt = \frac{(1 - x^2)^{\frac{1}{2}}\,dx + x^2\,(1 - x^2)^{-\frac{1}{2}}\,dx}{1 - x^2} = \frac{dx}{(1 - x^2)^{3/2}}. \qquad (1)$$

Now $$1 + t^2 = 1 + \frac{x^2}{1 - x^2} = \frac{1}{1 - x^2}. \qquad (2)$$

Dividing (1) by (2), the result is

$$\frac{dt}{1 + t^2} = \frac{dx}{(1 - x^2)^{1/2}} = dz,$$

or $$dz = (1 + t^2)^{-1}\, dt,$$

the rate of an arc of a circle in terms of the tangent and its rate.

Developing, $dz = (1 - t^2 + t^4 - t^6 + \text{etc.})\, dt$

and integrating, $$z = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \text{etc.}, \qquad (3)$$

which needs no correction, since $z = 0$ when $t = 0$.

Now, by means of the trigonometrical formula

$$\tan 2a = \frac{2 \tan a}{1 - \tan^2 a};$$

when $\tan a = \frac{1}{5}$, we find

$$\tan 4a = \frac{120}{119}.$$

Also, by means of the formula

$$\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B},$$

when $\tan A = \frac{120}{119}$ and $\tan B = \tan 45° = 1$, we find

$$\tan (A - B) = \frac{1}{239}.$$

Hence, four times the arc whose tangent is $\frac{1}{5}$ exceeds the arc of 45° by an arc whose tangent is $\frac{1}{239}$. In a similar manner, we shall find that twice the arc whose tangent is $\frac{1}{10}$ exceeds the arc whose tangent is $\frac{1}{5}$ by an arc whose tangent is $\frac{1}{515}$.

Therefore, if $z =$ arc of 45°, since (as has been shown in the paragraph immediately preceding)

$$\text{arc } 45° = 8 \tan \frac{1}{10} - 4 \tan \frac{1}{515} - \tan \frac{1}{239},$$

by applying these values to (3) and multiplying by 4, since arc 180° $= \pi$, the following are obtained:

$$\pi = 4 \left\{ \begin{array}{l} 8 \left(\frac{1}{10} - \frac{1}{3(10)^3} + \frac{1}{5(10)^5} - \frac{1}{7(10)^7} + \text{etc.} \right) \\ - 4 \left(\frac{1}{515} - \frac{1}{3(515)^3} + \frac{1}{5(515)^5} - \frac{1}{7(515)^7} + \text{etc.} \right) \\ - \left(\frac{1}{239} - \frac{1}{3(239)^3} + \frac{1}{5(239)^5} - \frac{1}{7(239)^7} + \text{etc.} \right) \end{array} \right\}$$

Six terms of the first line and three each of the second and third will give

$$\pi = 3.141592653589793.$$

The transcendental equation of the cycloid is (see Art. 42)

$$du = \frac{ydy}{(2ry - y^2)^{\frac{1}{2}}}.$$

Squaring this equation and substituting the value of dx^2 in the rate of the arc give

$$dz = \left(dy^2 + \frac{y^2 dy^2}{2ry - y^2}\right)^{\frac{1}{2}},$$

or, reducing, $$dz = dy \left(\frac{2r}{2r - y}\right)^{\frac{1}{2}}.$$

Putting this under the form

$$dz = (2r)^{\frac{1}{2}} (2r - y)^{-\frac{1}{2}} dy,$$

and integrating by Art. 74,

$$z = -2(2r)^{\frac{1}{2}} (2r - y)^{\frac{1}{2}} + C,$$

or $$z = -2\sqrt{\{2r\,(2r - y)\}} + C.$$

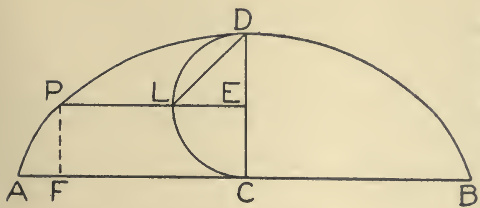


Fig 50

Estimating the arc from A, $z=0$ for $y=0$, consequently $0=-4r+C$ or $C=4r$, hence $z=4r-2\sqrt{\{2r(2r-y)\}}$, which represents the length of an arc of the cycloid, estimated from A to any point, as P.

If $$y=CD=2r$$
then $$z=APD=4r$$
and since $APD-AP=DP=4r-[4r-2\sqrt{\{2r(2r-y)\}}]$,
$$DP=2\sqrt{\{2r\,(2r-y)\}} \qquad (4)$$
which represents the length of the arc estimated from D to any point P.

By similar triangles,
$$CD:DL::DL:DE$$
or $$DL=(CD\cdot DE)^{1/2};$$
but, since $CD=2r$ and $y=PF=CE$, $DE=2r-y$, hence
$$DL=\sqrt{\{2r\,(2r-y)\}}$$
therefore, comparing this with (4), it will be found that
$$\text{arc}\,DP=2DL;$$
that is, the arc of the cycloid, estimated from the vertex of the axis CD, is equal to twice the corresponding chord DL of the generating circle.

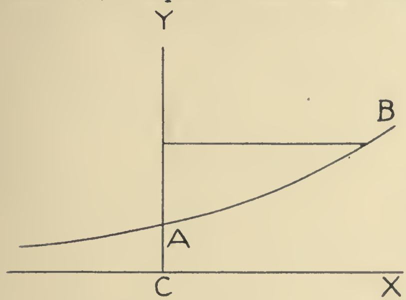


Fig. 51

The equation of the logarithmic curve is
$$x=\log y.$$

Passing to the rate and squaring,
$$dx^2=\frac{dy^2}{y^2}.$$

Substituting this value of dx^2 in the rate of the arc and reducing give

$$dz = \frac{(1+y^2)^{\frac{1}{2}}\,dy}{y}.$$

Integrating by formula C of Art. 80,

$$z = (1+y^2)^{\frac{1}{2}} + \int \frac{dy}{y\,(1+y^2)^{\frac{1}{2}}}.$$

Integrating again by Art. 83,

$$z = (1+y^2)^{\frac{1}{2}} - \log \frac{(1+y^2)^{\frac{1}{2}} + (1+y)}{(1+y^2)^{\frac{1}{2}} - (1-y)} + C;$$

or, multiplying both numerator and denominator of the fraction in the second member by $(1+y^2)^{\frac{1}{2}} + (1-y)$ and reducing,

$$z = (1+y^2)^{\frac{1}{2}} - \log \frac{1 + (1+y^2)^{\frac{1}{2}}}{y} + C.$$

With C the origin of coördinates, when $x=0$, $y=1$ and $z=0$; consequently

$$0 = \sqrt{2} - \log(1+\sqrt{2}) + C,$$

or $$C = -\sqrt{2} + \log(1+\sqrt{2}),$$

therefore

$$z = (1+y^2)^{\frac{1}{2}} - \log \frac{1 + (1+y^2)^{\frac{1}{2}}}{y} - \sqrt{2} + \log(1+\sqrt{2}),$$

which represents an arc of the logarithmic curve AB, estimated toward B from the point where it cuts the axis of coördinates.

The equation of the spiral of Archimedes is

$$r = \frac{v}{2\pi}.$$

Taking the rate and squaring,

$$dr^2 = \frac{dv^2}{4\pi^2}.$$

Substituting this value of dr in the rate of the curve and reducing,

$$dz = \frac{dv}{2\pi}(1+v^2)^{\frac{1}{2}}.$$

First integrating by formula C of Art. 80, then by Art. 83, (6),

$$z = \frac{1}{4\pi} [v (1 + v^2)^{\frac{1}{2}} - \log \{(1 + v^2)^{\frac{1}{2}} - v\}]$$

which represents the length of any arc of the spiral of Archimedes, estimated from the pole; no correction is needed, since $z = 0$ when $v = 0$.

EXAMPLES

1. Determine the length of an arc of the common parabola.

2. Determine the length of an elliptic quadrant in terms of its eccentricity, the semi-major axis being unity and the semi-minor axis a.

3. Determine the length of an arc of the logarithmic spiral, estimated from the point where $r = 1$.

Area of Curves

94. The rate of the area of a curve referred to rectangular coördinates is, by Art. 47,

$$dA = ydx,$$

which can be integrated when the second member is given in terms of y and dy, or x and dx.

The rate of the area of a polar curve is, by Art. 57,

$$dA = \frac{1}{2} r^2 dv,$$

which can be integrated when the second member is expressed in terms of r and dr, or v and dv.

Multiplying both members of the equation of the circle by dx gives

$$ydx = (R^2 - x^2)^{\frac{1}{2}} dx,$$

hence $$dA = (R^2 - x^2)^{\frac{1}{2}} dx.$$

Integrating, first by formula C of Art. 80, then by Art. 76,

$$A = \frac{1}{2} x (R^2 - x^2)^{\frac{1}{2}} + \frac{R^2}{2} \sin^{-1} \frac{x}{R}, \qquad (a)$$

which requires no correction, since $A = 0$ when $x = 0$.

Making $x = R$, since the arc of sine unity is $\frac{1}{2}\pi$,

$$A = \frac{1}{2} R^2 \pi,$$

which gives the area of a quadrant of a circle whose radius is R; therefore the area of the entire circle is $R^2\pi$.

The equation of the ellipse, referred to its center and axis, when both members are multiplied by dx, is

$$ydx = \frac{b}{a}(a^2 - x^2)^{1/2}\, dx;$$

hence $$dA = \frac{b}{a}(a^2 - x^2)^{1/2}\, dx.$$

Integrating, first by formula C of Art. 80, then by Art. 76,

$$A = \frac{b}{a}\{\frac{1}{2}x\,(a^2 - x^2)^{1/2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\}, \qquad (b)$$

which requires no correction, since $A = 0$ when $x = 0$.

If $x = a$, since the arc of sine unity is equal to $\frac{1}{2}\pi$, then

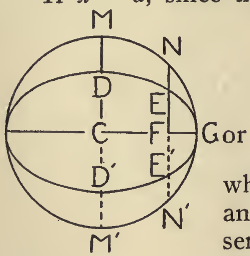


Fig. 52

$$A = \frac{b}{a}(\frac{1}{4}a^2\pi),$$

$$A = \frac{1}{4}ab\,\pi,$$

which represents the area of a quarter of an ellipse whose semi-major axis is a and semi-minor axis is b; therefore the area of the entire ellipse is equal to $ab\,\pi$.

Comparing (a) with (b), it will be seen that the area of a segment of the ellipse, as $CDEF$, is equal to the area of the corresponding segment of the circumscribing circle, $CMNF$, multiplied by $\frac{b}{a}$; hence

$$\text{area } DEE'D' = \frac{b}{a}(\text{area } MNN'M').$$

Taking the general equation of the parabola

$$y^n = ax \text{ or } y = a^{1/n}x^{1/n},$$

and multiplying both members by dx, the result is

$$ydx = a^{1/n}x^{1/n}dx;$$

hence $$dA = a^{1/n}x^{1/n}dx.$$

Integrating, $$A = \frac{n}{n+1}a^{1/n}x^{(n+1)/n} + C.$$

Estimating the area from the vertex of the parabola, $A=0$ when $x=0$, and consequently $C=0$; therefore

$$A=\frac{n}{n+1}a^{1/n}x^{(n+1)/n}=\frac{n}{n+1}x\left(a^{1/n}x^{1/n}\right),$$

or, substituting y for $a^{1/n}x^{1/n}$,

$$A=\frac{n}{n+1}xy, \tag{1}$$

which represents the area of a segment of any parabola, and is equal to the rectangle described by the abscissa and ordinate, multiplied by the constant term $\frac{n}{n+1}$.

If $n=2$, (1) becomes

$$A=\frac{2}{3}xy;$$

that is, the area of a segment of the common parabola is equal to two-thirds of the area of the rectangle described by the abscissa and ordinate.

If $n=1$, (1) becomes

$$A=\frac{1}{2}xy;$$

that is, the area of a triangle is equal to half the product of its base and perpendicular.

Multiplying both members of the equation of the hyperbola, referred to its center and axes, by dx gives

$$ydx=\frac{b}{a}(x^2-a^2)^{1/2}dx \text{ or } dA=\frac{b}{a}(x^2-a^2)^{1/2}dx.$$

Integrating first by formula C of Art. 80, then by Art. 83,

$$A=\frac{bx(x^2-a^2)^{1/2}}{2a}-\frac{ab}{2}\log\{x+(x^2-a^2)^{1/2}\}+C.$$

When $A=0$, $x=a$; consequently

$$C=-\frac{ab\log a}{2};$$

therefore

$$A=\frac{bx(x^2-a^2)^{1/2}}{2a}-\frac{ab}{2}\log\left\{\frac{x+(x^2-a^2)^{1/2}}{a}\right\},$$

or, since

$$\frac{b}{a}(x^2 - a^2)^{1/2} = y,$$

$$A = \frac{1}{2}xy - \frac{ab}{2}\log\left(\frac{bx + ay}{ab}\right).$$

Squaring both members of the equation of the spiral of Archimedes $(r = \frac{v}{2\pi})$ and multiplying by $\frac{1}{2}dv$ give

$$\frac{1}{2}r^2dv = \frac{v^2dv}{8\pi^2},$$

or

$$dA = \frac{v^2dv}{8\pi^2};$$

whence, integrating,

$$A = \frac{v^3}{24\pi^2} + C.$$

Estimating the area from the pole, $A = 0$ when $v = 0$, and consequently $C = 0$; therefore

$$A = \frac{v^3}{24\pi^2}.$$

If $v = 2\pi$, then

$$A = \frac{1}{3}\pi,$$

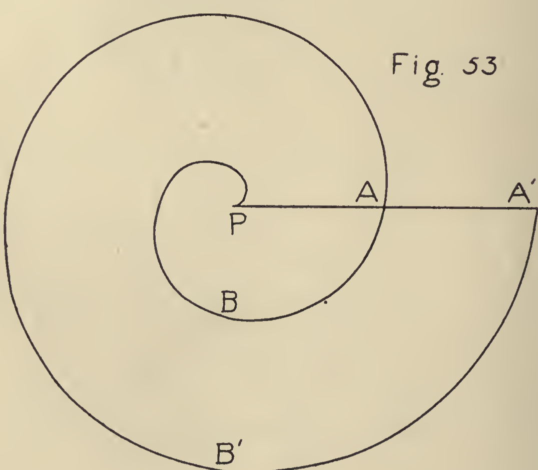


Fig. 53

which represents the area of PBA, described by one revolution of the radius vector: that is, the area of the first spire is equal to one-third of the area of a circle, whose radius is equal to the radius vector of the spiral at the end of the first revolution.

If $v = 4\pi$, then

$$A = \frac{8}{3}\pi,$$

which represents the area described by the radius vector in two revolutions; but it will be seen that the radius vector describes the portion PBA a second time; therefore, to obtain the area of $PB'A'$, the area described by the first revolution must be deducted: that is,

$$\text{area } PBA = \frac{8}{3}\pi - \frac{1}{3}\pi = \frac{7}{3}\pi.$$

EXAMPLES

1. Determine the area of the cycloid.

2. Determine the area of a segment of the logarithmic curve, lying between the curve and axis of ordinates, estimated from the point where the curve cuts the axis of ordinates.

Surface Areas of Revolution

95. For a curve referred to rectangular coördinates, revolving about the axis of X, the rate of the surface area of rotation is (see Art. 48)

$$dS = 2\pi y dz.$$

In case the curve is revolved about the axis of Y, it is evident that the rate of the surface area will then be

$$dS = 2\pi x dz.$$

When the second member of either of these equations is expressed in terms of x and dx or y and dy, the integral thereof may be determined.

From the equation of the common parabola it will be found that

$$dx = \frac{ydy}{p}.$$

Substituting this value of dx in the rate of the area of the surface of revolution,

$$dS = 2\pi y \left(\frac{y^2dy^2}{p^2} + dy^2\right)^{\frac{1}{2}},$$

or $$dS = \frac{2\pi y}{p}(y^2 + p^2)^{\frac{1}{2}} dy.$$

Integrating by Art. 78,

$$S = \frac{2\pi}{3p}(y^2 + p^2)^{3/2} + C.$$

Estimating the arc from the origin of coördinates, $S = 0$ when $y = 0$; hence $C = -\frac{2p^2\pi}{3}$ and

$$S = \frac{2\pi}{3p}\{(y^2 + p^2)^{3/2} - p^3\},$$

which represents the surface area of revolution of the common parabola for any ordinate y.

The equation of the ellipse is

$$a^2y^2 = a^2b^2 - b^2x^2,$$

$$ydz = \frac{b}{a}(a^2 - \frac{a^2 - b^2}{a^2}x^2)^{\frac{1}{2}} dx$$

or, representing the eccentricity of the ellipse by e, by substituting a^2e^2 for $a^2 - b^2$, since $a^2 - b^2 = a^2e^2$,

$$ydz = \frac{b}{a}(a^2 - e^2x^2)^{\frac{1}{2}} dx;$$

therefore $$dS = \frac{2be\pi}{a}(\frac{a^2}{e^2} - x^2)^{\frac{1}{2}} dx.$$

Integrating, first by formula C of Art. 80, then by Art. 76, gives

$$S = \frac{be\pi}{a}(\frac{a^2}{e^2} - x^2)^{\frac{1}{2}} x + \frac{ab\pi}{e}\sin^{-1}\frac{ex}{a},$$

which needs no correction, since $S = 0$ when $x = 0$; hence the expression represents the surface area of that part of an ellipsoid estimated from the vertex of the minor axis and corresponding to the abscissa x, the arc being revolved about the major axis. By making $x = a$ and reducing,

$$S = b^2\pi + \frac{ab\pi}{e}\sin^{-1} e,$$

which gives one-half the area of the surface of the ellipsoid;

therefore if S' represents the area of the entire surface, then

$$S' = 2b^2\pi + \frac{2ab\pi}{e} \sin^{-1} e.$$

When $a = b$, $e = 0$, and the equation becomes

$$S' = 2b^2\pi + 2b^2\pi = 4b^2\pi,$$

the area of the surface of a sphere whose semi-diameter is b.

If the ellipse be revolved about its minor axis, then will

$$xdz = \frac{a}{b^2} (b^4 + a^2e^2y^2)^{\frac{1}{2}} dy;$$

hence $$dS = \frac{2a\pi}{b^2} (b^4 + a^2e^2y^2)^{\frac{1}{2}} dy.$$

Integrating, first by formula C of Art. 80, then by Art. 83, gives

$$S = \frac{a\pi}{b^2} (b^4 + a^2e^2y^2)^{\frac{1}{2}} y +$$

$$\frac{b^2\pi}{e} \log \{ (b^4 + a^2e^2y^2)^{\frac{1}{2}} + aey \} + C.$$

Estimating the surface from the vertex of the major axis, $S = 0$ when $y = 0$, in which case

$$C = -\frac{b^2\pi}{e} \log b^2;$$

therefore $$S = \frac{a\pi}{b^2} (b^4 + a^2e^2y^2)^{\frac{1}{2}} y +$$

$$\frac{b^2\pi}{e} \log \{ (b^4 + a^2e^2y^2)^{\frac{1}{2}} + aey \} - \frac{b^2\pi}{e} \log b^2,$$

or $$S = \frac{a\pi}{b^2} (b^4 + a^2e^2y^2)^{\frac{1}{2}} y + \frac{b^2\pi}{e} \log \left\{ \frac{(b^4 + a^2e^2y^2)^{\frac{1}{2}} + aey}{b^2} \right\}.$$

Now, since $b = a (1 - e^2)^{\frac{1}{2}}$ and $(b^2 + a^2e^2)^{\frac{1}{2}} = a$, when $y = b$, this becomes

$$S = a^2\pi + \frac{b^2\pi}{e} \log \frac{ab (1 + e)^{\frac{1}{2}}}{ab (1 - e)^{\frac{1}{2}}}$$

or $$S = a^2\pi + \frac{b^2\pi}{e} \log \frac{(1 + e)^{\frac{1}{2}}}{(1 - e)^{\frac{1}{2}}},$$

which represents half the area of the surface of a spheroid. If S' represents the entire surface, then

$$S' = 2a^2\pi + 2b^2\pi\{\frac{\log(1+e)^{\frac{1}{2}} - \log(1-e)^{\frac{1}{2}}}{e}\}.$$

If $a = b$, then

$$S' = 2b^2\pi\{1 + \frac{\log(1+e)^{\frac{1}{2}} - \log(1-e)^{\frac{1}{2}}}{e}\}.$$

Now, when $a = b$, $e = 0$; but by Art. 35,

$$\frac{\log(1+e)^{\frac{1}{2}} - \log(1-e)^{\frac{1}{2}}}{e} = 1;$$

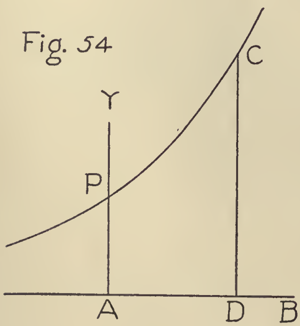
therefore the surface of a sphere whose semi-diameter is b, is

$$S' = 4b^2\pi.$$

From the equation of the logarithmic curve, it will be found that

$$ydz = (1+y^2)^{\frac{1}{2}}dy, \text{ hence } dS = 2\pi(1+y^2)^{\frac{1}{2}}dy.$$

Fig. 54



Integrating by Arts. 80 and 83,

$$S = \pi\{(1+y^2)^{\frac{1}{2}}y + \log[(1+y^2)^{\frac{1}{2}} + y]\} + C.$$

Estimating the surface from P, the point where the axis of ordinates cuts the curve, $S = 0$ when $y = 1$; consequently

$$C = \pi\{-\sqrt{2} - \log(\sqrt{2}+1)\};$$

therefore

$$S = \pi\{y(1+y^2)^{\frac{1}{2}} + \log[(1+y^2)^{\frac{1}{2}} + y]\} - \pi\{\sqrt{2} + \log(\sqrt{2}+1)\}.$$

This represents the area of the surface of revolution of any arc of the logarithmic curve, estimated from P, as PC, and corresponding to the ordinate $y = DC$, the curve being revolved about the axis of abscissas AB.

EXAMPLES

1. Determine the area of the convex surface of a right conoid whose perpendicular height is a and diameter of base is b.

2. Determine the area of surface of revolution of the cycloid when revolved about its base.

3. Determine the area of the convex surface of a cubical paraboloid, when the axis of ordinates is the axis of revolution. The equation is $y^3 = ax$.

Volume of Revolution

96. The rate of the volume of revolution generated by an arc of a curve revolved about its axis of abscissas is, by Art. 49,

$$dV = \pi y^2 dx. \qquad (1)$$

When the second member of this equation is expressed in terms of either x and dx, or y and dy, its integral can be determined.

When the axis of Y is the axis of revolution, the rate of the volume of revolution thus generated is

$$dV = \pi x^2 dy.$$

From the general equation of the parabola, $y^n = ax$, it will be found that

$$dx = \frac{n}{a} y^{n-1} dy.$$

Substituting this value of dx in (1),

$$dV = \frac{n}{a} \pi y^{n+1} dy,$$

and integrating, the result is

$$V = \frac{n}{a(n+2)} \pi y^{n+2} + C = \pi y^2 \left(\frac{n}{n+2} \cdot \frac{y^n}{a}\right) + C,$$

or since $\dfrac{y^n}{a} = x$, $\quad V = \pi y^2 \left(\dfrac{n}{n+2} x\right) + C,$

which needs no correction, since $v = 0$ when $x = 0$. If $n = 1$,

then $$V = \frac{1}{3} \pi y^2 x,$$

which represents the volume of a right cone whose altitude is x and y one-half of the diameter of its base.

If $n = 2$, then $$V = \frac{1}{2} \pi y^2 x,$$

which represents the volume of the common parabola.

The equation of the ellipse, when the origin is at the vertex of its minor axis, is

$$x^2 = \frac{a^2}{b^2}(2by - y^2);$$

hence [see (2)],

$$dV = \frac{a^2\pi}{b^2}(2by - y^2)\,dy.$$

Integrating, $V = \frac{a^2\pi}{b^2}(by^2 - \frac{1}{3}y^3) + C,$

in which $C = 0$ when $y = 0$; therefore

$$V = \frac{a^2\pi}{b^2}(by^2 - \frac{1}{3}y^3).$$

If $y = b$, then

$$V = \frac{2}{3}a^2b\,\pi,$$

which represents the volume of one-half of a spheroid; hence the entire volume is

$$V' = \frac{4}{3}a^2b\,\pi = \frac{2}{3}(2a^2b\,\pi).$$

But $2a^2b\,\pi$ represents the volume of a cylinder, whose altitude is $2b$ and the radius of whose base is a; therefore the volume of a spheroid is equal to two-thirds of the volume of a circumscribed cylinder.

The equation of the hyperbola, when the origin of the coördinates is at the vertex of the transverse axis, is

$$y^2 = \frac{b^2}{a^2}(x^2 + 2ax).$$

Substituting this value of y^2 in (1),

$$dV = \frac{b^2\pi}{a^2}(x^2 + 2ax)\,dx.$$

Integrating, $V = \frac{b^2\pi}{a^2}(\frac{1}{3}x^3 + ax^2) + C.$

Estimating the volume of revolution from the origin of coördinates, we have $V = 0$ when $x = 0$, and consequently $C = 0$; therefore

$$V = \frac{b^2\pi}{a^2}\left(\frac{1}{3}x^3 + ax^2\right),$$

which represents the volume of revolution of the hyperbola for any abscissa.

The ratal equation of the cycloid is

$$dx = \frac{ydy}{(2ry - y^2)^{½}}.$$

Substituting this value of dx in (1),

$$dV = \frac{\pi y^3 dy}{(2ry - y^2)^{½}},$$

the integral of which is

$$V = \pi\left\{\frac{1}{6}(2y^2 + 5ry + 15r^2)(2ry - y^2)^{½} + \frac{5}{2}r^3 \text{ vers}^{-1}\frac{y}{r}\right\}.$$

If $y = 2r$, then $V = \frac{5\pi}{2} r^3 \text{ vers}^{-1} 2,$

or, since $\text{vers}^{-1} 2 = \pi$,

$$V = \frac{5}{2} r^3\pi^2,$$

which represents one-half the volume of revolution generated by the cycloid revolved about its base. The entire volume is

$$V' = 5r^3\pi^2.$$

EXAMPLES

1. Determine the volume of rotation of the ellipse when the origin of the coördinates is at the vertex of the major axis.

2. Determine the volume of revolution of the logarithmic curve when revolved about its axis of abscissas.

3. Determine the volume of revolution about its axis of abscissas of the curve whose equation is

$$y = x(x + a).$$

97. Let $BDEF$ be a plane moving from A toward X along the axis of X and at right-angles thereto, and let AC be repre-

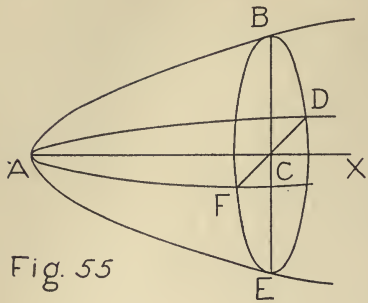


Fig. 55

sented by x, BC by y, and FC by v; then the rate of the volume of the solid thus generated will be

$$dV = f(v, y)\, dx,$$

or, since $y = f(x)$ and $v = f(y)$,

$$dV = f(x)\, dx \qquad (1)$$

This formula is applicable to the volume of any solid, when the area of the plane $BDEF$ can be expressed in terms of x and dx.

Determine the volume of a right pyramid whose base is a rectangle.

Let the perpendicular Aa be represented by x, the side BC by y, and the side CD by v; also let $y = ax$ and $v = by$. Then the area of $BCDE$ will be $vy = abx^2$; hence

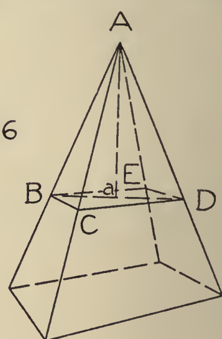
$$dV = abx^2 dx,$$

and integrating,

$$V = \frac{1}{3} abx^3.$$

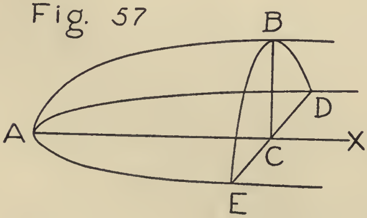
But abx^2 is the area of $BCDE$; therefore the volume of the pyramid is equal to the area of its base multiplied by one-third of its perpendicular height.

Fig. 56



Required the volume of a parabolic paraboloid, the fixed parabola being the semi-cubical and the generatrix the common parabola.

Fig. 57



Let $x = AC$, $y = BC$, and $v = CE = CD$; then for ABX,

$$y^{3/2} = ax,$$

and for BCE, $v^2 = by$.

From these equations it will be found that the area of

BDE is $\frac{4}{3} ab^{1/2} x$,

also $$dV = \frac{4}{3} ab^{1/2} x dx,$$

and integrating, $$V = \frac{2}{3} ab^{1/2} x^2,$$

Required the volume of an elliptical ellipsoid, the equations being for the fixed ellipse $a^2y^2 = a^2b^2 - b^2x^2$ and for the generatrix $a^2v^2 = a^2c^2 - c^2x^2$ and the origin of coördinates being at the center.

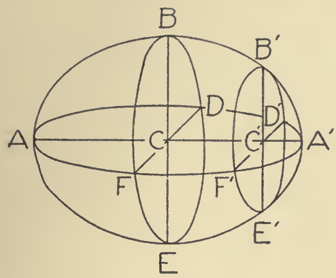


Fig. 58

Let $A'C = a$, $BC = b$, $CF = c$, $CC' = x$, $B'C' = y$, and $C'F' = v$; then from the equations,

$$y = \frac{b}{a}(a^2 - x^2)^{1/2},$$

$$v = \frac{c}{a}(a^2 - x^2)^{1/2},$$

and $$vy = \frac{bc}{a^2}(a^2 - x^2).$$

But $\pi vy =$ area of $B'D'E'F'$, therefore

$$dV = \frac{\pi bc}{a^2}(a^2 - x^2)\,dx = (\pi bc - \frac{\pi bc}{a^2}x^2)\,dx,$$

and integrating, $$V = \pi bcx - \frac{\pi bc}{3a^2}x^3,$$

which requires no correction since $V = 0$ when $x = 0$; hence, making $x = a$,

$$V = \pi abc - \frac{1}{3}\pi abc = \frac{2}{3}\pi abc.$$

Since V is one-half the volume of the ellipsoid, V', the volume of the entire solid, is $\frac{4}{3}\pi abc$.

EXAMPLES

1. Determine the volume of an elliptical conoid whose altitude is a' and the radius of whose base is b'.

2. Determine the volume of a groin formed by the intersection of two equal semi-cylinders at right-angles to each other, the equation being $y = 2rx - x$.

Curved Surfaces Referred to Three Rectangular Coördinates

98. To obtain a formula for the volume of a solid bounded by a curved surface and referred to three rectangular coördinates, x, y, and z, of which $z = f(x, y)$.

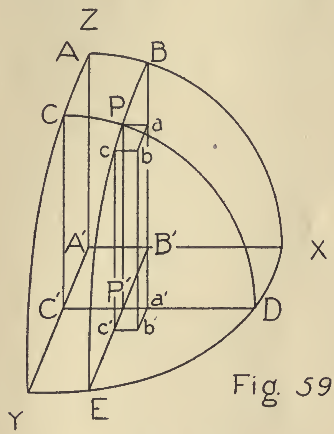


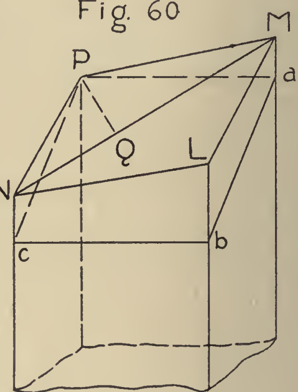
Fig. 59

Let the plane $C'CPD$ be parallel to the plane $A'ZX$ and the plane $EPBB'$ parallel to the plane YZA'; also let $A'B' = C'P' = x$, $A'C' = B'P' = y$ and $P'P = z$. Represent dx by $P'a' = c'b'$ and dy by $P'c' = a'b'$; then $zdxdy$ will represent the rate of the volume of the solid; that is,

$$d^2V = zdxdy. \qquad (A)$$

To obtain a formula for the surface area, let $Pa = cb$ represent dx, $Pc = ab$ represent dy, and $Nc = Ma$ represent dz; also let PM be a tangent to the curve CPD at P (Fig. 60), PN a tangent to the curve BPE at P, and PQ a perpendicular to NM. Then will

Fig. 60



$$PN = (dx^2 + dz^2)^{\frac{1}{2}},$$
$$PM = (dy^2 + dz^2)^{\frac{1}{2}},$$
and $$NM = (dx^2 + dy^2)^{\frac{1}{2}}.$$

From these three equations the following is obtained:

$$PQ = \frac{(dx^2dy^2 + dx^2dz^2 + dy^2dz^2)^{\frac{1}{2}}}{(dx^2 + dy^2)^{\frac{1}{2}}};$$

but $PQ \cdot NM =$ area of $NPML =$

$$(dx^2dy^2 + dx^2dz^2 + dy^2dz^2)^{\frac{1}{2}};$$

therefore $d^2S =$

$$(dx^2dy^2 + dx^2dz^2 + dy^2dz^2)^{\frac{1}{2}}. \quad (B)$$

Required the volume, also the surface area of a sphere, the equation being

$$z^2 = r^2 - (x^2 + y^2). \qquad (1)$$

For the volume [see formula (A)], it will be seen that

$$d^2V = (r^2 - x^2 - y^2)^{\frac{1}{2}} dxdy = zdxdy.$$

The integral of this with respect to y, between the limits $y=0$ and $y=(r^2-x^2)^{½}$, is

$$dV=\frac{1}{2}(r^2-x^2)dx\sin^{-1}\{\frac{y}{(r^2-x^2)^{½}}\}=\frac{1}{4}\pi(r^2-x^2)dx,$$

since $(r^2-x^2)^{½}=y$ and $\sin^{-1}\frac{y}{y}=\frac{1}{2}\pi$.

Integrating this expression with respect to x gives

$$V=\frac{1}{4}\pi\,(r^2x-\frac{1}{3}x^3)+C,$$

or between the limits $x=0$ and $x=r$,

$$V=\frac{1}{4}\pi\,(r^3-\frac{1}{3}r^3)=\frac{1}{6}r^3\,\pi,$$

which represents one-eighth of the volume of a sphere; therefore the volume of the entire sphere is

$$V'=\frac{4}{3}\pi r^3.$$

Resuming (1),

$$z^2=r^2-x^2-y^2,$$

and taking the rate, first with respect to x, then with respect to y, the results are

$$dz=-\frac{xdx}{z} \text{ and } dz=-\frac{ydy}{z};$$

hence $$dz^2=\frac{y^2dy^2}{z^2} \text{ and } dz^2=\frac{x^2dx^2}{z^2}.$$

Substituting these values of dz^2 in formula B, so that

$$\frac{dx^2dz^2}{z^2}+\frac{dy^2dz^2}{z^2}$$

shall read $$\frac{x^2dx^2dy^2}{z^2}+\frac{y^2dy^2dx^2}{z^2}$$

and reducing (since dz will be eliminated), the result is

$$d^2S=\frac{dxdy}{z}\,(x^2+y^2+z^2)^{½},$$

or, since $$(x^2 + y^2 + z^2)^{\frac{1}{2}} = r,$$

and $$z = (r^2 - x^2 - y^2)^{\frac{1}{2}},$$

$$d^2S = \frac{rdxdy}{(r^2 - x^2 - y^2)^{\frac{1}{2}}}.$$

The integral of this with respect to y, between the limits of $y = 0$ and $y = (r - x)$, is

$$dS = r \sin^{-1} \frac{ydx}{(r^2 - x^2)^{\frac{1}{2}}}.$$

$$y = (r^2 - x^2)^{\frac{1}{2}}, \quad \frac{y}{(r^2 - x^2)^{\frac{1}{2}}} = 1;$$

therefore, since $\sin^{-1} 1 = \frac{1}{2}\pi$,

$$dS = \frac{1}{2} r \pi \, dx,$$

the integral of which, between the limits of $x = 0$ and $x = r$, is

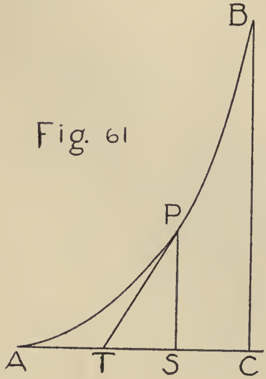
$$S = \frac{1}{2} r^2 \pi,$$

which represents one-eighth of the surface area of a sphere, therefore the entire area is

$$S' = 4r^2\pi.$$

99. A body T, with a uniform velocity, proceeds from C, toward A, along the straight line CA, and a body P, with a velocity which is to that of T as 1 to n, proceeds from B in pursuit of T and always in the direction of T. Required the equation of the curve APB, called the curve of pursuit, which is described by P.

Fig. 61



Let A be the origin, $BC = a$, $AS = x$, $PS = y$, and the arc $AP = z$; then $AT = nz$ and (see Art. 40) the

subtangent $ST = \frac{ydx}{dy}$.

Now, since

$$AT = AS - ST = x - \frac{ydx}{dy},$$

$$x - \frac{ydx}{dy} = nz.$$

Taking the rate of this, regarding dy as constant, and reducing give

$$-\frac{yd^2x}{dy} = ndz; \qquad (1)$$

but $$dz = (dx^2 + dy^2)^{1/2} = dy\,(\frac{dx^2}{dy^2} + 1)^{1/2}.$$

From this and (1), the following is found:

$$-\frac{ndy}{y} = (\frac{dx^2}{dy^2} + 1)^{-1/2}\frac{d^2x}{dy}.$$

Integrating by Art. 84, regarding $\frac{dx}{dy}$ as the variable, the result is

$$-n \log y = \log \{\frac{dx}{dy} + (\frac{dx^2}{dy^2} + 1)^{1/2}\} + C.$$

Since $\frac{dx}{dy} = \tan SPT$, when $\frac{dx}{dy} = 0$, $y = a$; therefore

$$-n \log a = C,$$

hence, transposing the value of C, it is found that

$$n \log a - n \log y = \log \{\frac{dx}{dy} + (\frac{dx^2}{dy^2} + 1)^{1/2}\},$$

or $$\log \frac{a^n}{y^n} = \log \{\frac{dx}{dy} + (\frac{dx^2}{dy^2} + 1)^{1/2}\};$$

hence $$\frac{a^n}{y^n} = \frac{dx}{dy} + (\frac{dx^2}{dy^2} + 1)^{1/2}.$$

By resolving this, it will be found that

$$dx = \frac{1}{2}a^n y^{-n} dy - \frac{1}{2}a^{-n} y^n dy, \qquad (2)$$

the integral of which is

$$x = \frac{a^n}{2(1-n)} y^{1-n} - \frac{1}{2a^n(1+n)} y^{1+n}, \qquad (3)$$

which needs no correction, since $y = 0$ when $x = 0$, and therefore is the required equation.

Dividing (3) by n, then, when $y = a$ and $x = AC$,

$$\frac{x}{n} = \frac{a}{2n(1-n)} - \frac{a}{2n(1+n)} = \frac{a}{1-n^2},$$

or, since $x = AC$ and $\frac{AC}{n} = APB$,

$$APB = \frac{a}{1-n^2}. \qquad (4)$$

When $n = \frac{1}{2}$, (3) and (4) become

$$x = \frac{y^{\frac{1}{2}}}{a^{\frac{1}{2}}} \left(a - \frac{1}{3} y\right),$$

and

$$APB = \frac{4}{3} a.$$

From (1), (2), and (3) it will be found that

$$\frac{1}{n}\left(x - y\frac{dx}{dy}\right) = \frac{a^n}{2(1-n)} y^{1-n} + \frac{1}{2a^n(1+n)} y^{1+n};$$

and

$$z = \frac{a^n}{2(1-n)} y^{1-n} + \frac{1}{2a^n(1+n)} y^{1+n}, \qquad (5)$$

which represents the length of any arc, as AP. When $n = \frac{1}{2}$, (5) becomes

$$z = a^{\frac{1}{2}} y^{\frac{1}{2}} + \frac{1}{3a^{\frac{1}{2}}} y^{3/2} = \frac{y^{\frac{1}{2}}}{3a^{\frac{1}{2}}}(3a + y).$$